QUASI-COMPACTNESS OF TRANSFER OPERATORS FOR CONTACT ANOSOV FLOWS

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Abstract. For any C^r contact Anosov flow with $r \geq 3$, we construct a scale of Hilbert spaces, which are embedded in the space of distributions on the phase space and contain all the C^r functions, such that the one-parameter family of the transfer operators for the flow extend to them boundedly and that the extensions are quasi-compact. Further we give explicit bounds on the essential spectral radii of those extensions in terms of the differentiability r and the hyperbolicity exponents of the flow.

1. Introduction

1.1. **Main result.** Geodesic flows on closed Riemannian manifolds with negative sectional curvature are a typical class of flows that exhibit chaotic behavior of orbits and have been studied extensively since the works of Hopf[15] and Anosov[3] for this reason. Ergodicity and mixingness, which characterize chaotic dynamical systems qualitatively, are established for those flows already in early stage of study[15, 3]. However, quantitative estimates on the rate of mixing were obtained only recently in late 90's, while there had been some precise results in the case of constant curvature by means of representation theory[10, 22, 24]. This is quite in contrast to the case of Anosov diffeomorphisms for which exponential decay of correlations had been established already in 70's[8]. The difficulty in the case of geodesic flows (or hyperbolic flows, more generally) is in brief that there is no exponential expansion nor contraction in the flow direction. The mechanism behind mixing in hyperbolic flows is different from and in fact subtler than that in hyperbolic discrete dynamical systems.

In 1998, Chenov[9] made a breakthrough by showing that the rate of mixing is streched exponential at slowest for 3-dimensional Anosov flows satisfying the uniform non-integrability condition and, in particular, for all geodesic flows on closed surfaces with negative variable curvature. Chernov also conjectured in [9] that the rate should be exponential. Shortly, this conjecture is proved affirmatively by Dolgopyat[11]. Dolgopyat analyzed the perturbed transfer operators closely and gave a necessary estimate on

Date: January 14, 2010.

2000 *Mathematics Subject Classification.* 37D20,37A25.

Key words and phrases. Anosov flow, Transfer operator, Decay of correlations.

This work is partly supported by KAKENHI (B) 18340044.

the Laplace transforms of the correlations. Dolgopyat's method has been extended and applied to many situations to get exponential or rapid decay of correlations. ([2, 6, 12, 13, 14, 21, 25, 26, 27, 28])

More recently, Liverani[20] established exponential decay of correlations for C^4 contact Anosov flows and, in particular, for C^4 geodesic flows on closed Riemannian manifolds with negative curvature in arbitrary dimension. He combined Dolgopyat's method with his method of using Banach spaces of distributions developped in his previous paper[7]. A remarkable feature of the argument in [20] is that it is free from Markov partitions, which was a convenient artifact used in many works including [9] and [11] and was an obstacle in making use of smoothness of the flow.

The aim of this paper is to proceed one step further along the line of study described above. For any C^r contact Anosov flow with $r \geq 3$, we construct a scale of Hilbert spaces, which are embedded in the space of distributions on the phase space and contain all the C^r functions, so that the one-parameter family of the transfer operators for the flow extend naturally to bounded operators on them and that the extensions are *quasi-compact*. Further we give explicit upper bound on the essential spectral radii of the extensions in terms of differentiability r and hyperbolicity exponents of the flow. This result yields not only exponential decay of correlations but also a precise asymptotic estimate on the decay rate. (See Corollary 1.2.) Also our argument is free from Markov partitions.

To state the main result more precisely, we introduce some definitions. Let $d \geq 1$ and $r \geq 3$ be integers. Let M be an orientable $(2d+1)$-dimensional closed C^r manifold and α a C^r contact form on M. By definition, α is a 1-form such that $\omega := \alpha \wedge (d\alpha)^d$ is a volume form on M. Let $F^t : M \to M$ be a C^r Anosov flow preserving the contact form α. Such a flow is called a C^r contact Anosov flow. Geodesic flows on closed Riemannian manifolds with negative sectional curvature are types of contact Anosov flows, regarded as flows on the unit cotangent bundles equipped with the canonical contact forms.

Let v be the vector field that generates the flow F^t. By the definition of Anosov flow, there exists an invariant splitting of the tangent bundle, $TM = E^c \oplus E^s \oplus E^u$, such that E^c is the one-dimensional subbundle spanned by the vector field v and that there exist $\lambda_0 > 0$ and $C > 0$ such that[(1)]

$$\|DF_z^t|_{E^s}\| \leq C \cdot 2^{-\lambda_0 t} \quad \text{and} \quad \|DF_z^{-t}|_{E^u}\| \leq C \cdot 2^{-\lambda_0 t} \quad \forall t \geq 0, \forall z \in M.$$

Since the flow F^t preserves the contact form α, the subspaces E^s and E^u should be contained in the null space of α. This implies that the subspace $E^s \oplus E^u$ coincides with the null space of α and hence that $\alpha(v) \neq 0$ at any point. In what follows, we suppose $\alpha(v) \equiv 1$ by replacing α by $\alpha/\alpha(v)$. Since the 2-form $d\alpha$ is preserved by the flow F^t and gives a symplectic form on the null space of α, we see that $\dim E^s = \dim E^u = d$ and also that E^0

[(1)] For convenience in the later argument, we consider the exponential function with base 2 in many places below, though it is of course not essential.

coincide with the null space of $d\alpha$ at each point. Notice that the vector field v is characterized by the conditions $\alpha(v) = 1$ and $v \in \mathrm{null}\, d\alpha$. It is called the Reeb vector field of α.

Let $\Lambda_0 > 0$ be anther constant for the flow F^t such that, for some $C > 1$,

$$|\det(DF_z^{-t}|_{E^u})| \le C \cdot 2^{-\Lambda_0 t} \qquad \forall t \ge 0, \forall z \in M.$$

Obviously we may take Λ_0 so that $\Lambda_0 \ge d\lambda_0$. For the flow F^t, we associate the one-parameter family of transfer operators $\mathcal{L}^t : C^r(M) \to C^r(M)$ defined by $\mathcal{L}^t(u)(z) = u \circ F^t(z)$. For a real number s with $|s| \le r$, let $W^s(M)$ be the Sobolev space(2) of order s on M. Our main result is the following spectral property of $\mathcal{L}^t$.

Theorem 1.1. *For each $0 < \beta < (r-1)/2$, there exists a Hilbert space B^β, which is contained in $W^s(M)$ for $s < -\beta$ and contains $W^s(M)$ for $s > \beta$, such that the transfer operator $\mathcal{L}^t$ for sufficiently large t extends to a bounded operator on B^β and the essential spectral radius of the extension $\mathcal{L}^t : B^\beta \to B^\beta$ is bounded by $\max\{2^{-\Lambda_0 t/2}, 2^{-\beta\lambda_0 t}\} < 1$.*

Since contact Anosov flows are mixing (or even Bernoulli[19]) with respect to the contact volume ω, Theorem 1.1 implies not only exponential decay of correlations but also the following asymptotic estimate on correlations. (See [30] for the detail of the deduction.)

Corollary 1.2. *For any $0 < \alpha < \min\{\Lambda_0, (r-1)\lambda_0\}/2$, there exists finitely many complex numbers η_i with $-\alpha \le \Re(\eta_i) < 0$ and integers $k_i \ge 0$ for $1 \le i \le \ell$ such that, for any ψ and φ in $C^r(M)$, we have the asymptotic estimate for the correlation*

$$\frac{1}{\omega(M)}\int \psi \cdot \varphi \circ F^t \, d\omega \;-\; \frac{1}{\omega(M)}\int \psi \, d\omega \cdot \frac{1}{\omega(M)}\int \varphi \, d\omega = \sum_{i=1}^{\ell}\sum_{j=0}^{k_i} C_{ij}(\varphi,\psi)\cdot t^j 2^{t\eta_i} + \mathcal{O}(2^{-\alpha t})$$

as $t \to \infty$, where $C_{ij}(\varphi,\psi)$ are constants depending on ψ and φ bilinearly.

Also we can deduce from Theorem 1.1 the central limit theorem and the (generalized) local limit theorem for observables in $C^r(M)$ by a general abstract argument. (See [18].)

1.2. **Plan of the proof.** In the following sections, we proceed to the proof of the main theorem as follows. Section 2, 3 and 4 are devoted to preliminary argument. In Section 2, we set up a finite system of local charts on M adapted to the contact structure α and the hyperbolic structure of the flow. In Section 3, we then reduce the main theorem to the corresponding claim (Theorem 3.2) about transfer operators on the local charts. This reduction indicates in particular that our argument is irrelevant to the global

(2)See Remark 3.1 for the definition. For $s \ge 0$, $W^s(M)$ contains $C^s(M)$, and $W^{-s}(M)$ is contained in the space of distributions of order s.

structure of the flow. In Section 4, we give a local geometric property of the diffeomorphisms between the local charts induced by the time-t-maps of the flow. This property is simple but crucial for our argument.

In Section 5 and 6, we define Hilbert spaces $\mathscr{B}^\beta_\nu$ for real numbers β and ν, which consist of distributions on the unit disk $\mathbb{D}$ in the Euclidean space E of dimension $2d+1$. The Hilbert spaces B^β in the main theorem is made up from copies of such Hilbert spaces on the local charts by using a partition of unity on M. In Section 5, we construct a C^∞ countable partition of unity $\{p_\gamma\}_{\gamma\in\Gamma}$ on the cotangent bundle $T^*_{\mathbb{D}}E = \mathbb{D}\times E^*$. Then, in Section 6, we give a method of decomposing a function u on $\mathbb{D}$ into countably many smooth components u_γ, $\gamma\in\Gamma$, by using the pseudodifferential operators with symbol p_γ. By definition, each component u_γ is a "*wave packet*" which are localized both in the real and frequency spaces. The Hilbert space $\mathscr{B}^\beta_\nu$ will defined as the completions of the space $C^\infty(\mathbb{D})$ of C^∞ functions on the unit disk $\mathbb{D}$ with respect to a norm $\|\cdot\|_{\beta,\nu}$ that counts the L^2 norms of the components u_γ with some appropriate weight.

Our basic strategy is that we regard a transfer operator $\mathcal{L}$ acting on $\mathscr{B}^\beta_\nu$ as an infinite matrix of operators $\mathcal{L}_{\gamma\gamma'}$, each of which concerns the transition from one component to another induced by $\mathcal{L}$ and deduce the required properties of $\mathcal{L}$ from relatively simple estimates on each $\mathcal{L}_{\gamma\gamma'}$. In Section 7, we introduce some definitions in order to describe the argument along this strategy. And we find that each operator $\mathcal{L}_{\gamma\gamma'}$ is a tame integral operator with smooth rapidly decaying kernel. Further we give simple estimates on the kernel of $\mathcal{L}_{\gamma\gamma'}$, regarding it as an oscillatory integral.

Section 8–12 are the main body of the proof. In the proof, we divide the transfer operator $\mathcal{L}$ on the local charts into three parts: the *compact*, *central* and *hyperbolic part*. The compact part is the part that concerns the components of functions with low frequencies. In Section 8, we show that the compact part is in fact a compact operator and therefore negligible in our argument because the essential spectral radius of an operator does not change by perturbation by compact operators. The definitions of the central and hyperbolic part are more involved. Roughly, the central part is the part that concerns the components of functions which are localized along the central (or flow) direction in the frequency space, and the hyperbolic part is the remainder.

In Section 9–11, we deal with the hyperbolic part and estimate its operator norm. The argument in these sections makes use of hyperbolicity of the flow in the directions transversal to the flow, and is partially similar to that in our previous paper [5, 4] on hyperbolic diffeomorphisms. The estimate on the hyperbolic part leads to the term $2^{-\beta\lambda_0 t}$ in the main theorem.

In Section 12, we deal with the central part, which is responsible for the difficulty in the case of hyperbolic flow noted in the beginning. The argument on the central part is in fact the main point of this paper and makes use of the non-integrability of the contact form α essentially. The estimate on the central part leads to the term $2^{-\Lambda_0 t/2}$ in the main theorem.

Remark 1.3. A prototype of the argument on the central part can be found in the author's previous paper [30], where a class of expanding semi-flows are considered as a simplified model of Anosov flows.

2. Darboux theorem for contact structure

In this section, we set up a finite system of coordinate charts on M which is adapted to the contact structure α on M and also to the hyperbolic structure of the flow F^t. Let E be an Euclidean space of dimension $2d+1$, equipped with an orthonormal coordinate

$$x = (x_0, x_1^+, \dots, x_d^+, x_1^-, \dots, x_d^-).$$

Let E^* be the dual space of E, equipped with the dual coordinate

$$\xi = (\xi_0, \xi_1^+, \dots, \xi_d^+, \xi_1^-, \dots, \xi_d^-),$$

so that evaluation of $\xi \in E^*$ at $x \in E$ is given by

$$\langle \xi, x\rangle = \xi_0 \cdot x_0 + \xi_1^+ \cdot x_1^+ + \cdots + \xi_d^+ \cdot x_d^+ + \xi_1^- \cdot x_1^- + \cdots + \xi_d^- \cdot x_d^-.$$

For brevity, we write $x = (x_0, x^+, x^-)$ and $\xi = (\xi_0, \xi^+, \xi^-)$ for x and ξ as above, setting $x^\pm = (x_1^\pm, \dots, x_d^\pm)$ and $\xi^\pm = (\xi_1^\pm, \dots, \xi_d^\pm)$ respectively. Let $E = E_0 \oplus E_+ \oplus E_-$ and $E^* = E_0^* \oplus E_+^* \oplus E_-^*$ be the corresponding orthogonal decomposition. For $\sigma \in \{0, +, -\}$, let $\pi_\sigma : E \to E_\sigma$ and $\pi_\sigma^* : E^* \to E_\sigma^*$ be the orthogonal projections. Also we set $\pi_{+,-} = \pi_+ \oplus \pi_- : E \to E_+ \oplus E_-$ and define $\pi_{0,+}$, $\pi_{0,-}$, $\pi_{+,-}^*$, $\pi_{0,+}^*$ and $\pi_{0,-}^*$ analogously.

The standard contact form on the Euclidean space E is the 1-form

$$\alpha_0 = dx_0 + x^- \cdot dx^+ - x^+ \cdot dx^-$$

where $x^- \cdot dx^+ = \sum_{i=1}^d x_i^- \cdot dx_i^+$ and $x^+ \cdot dx^- = \sum_{i=1}^d x_i^+ \cdot dx_i^-$. We will refer $v_0 = \partial/\partial x_0$ as the standard vector field on E, which is nothing but the Reeb vector field of α_0. A local chart $\kappa : U \to V \Subset E$ on an open subset $U \subset M$ is called a Darboux chart if $\kappa^*(\alpha_0) = \alpha$ on U. Darboux theorem for contact structure[1, pp.168] tells that there exists a system of Darboux charts on M.

Let $\mathbf{C}_+$ and $\mathbf{C}_-$ be the closed cones on E defined by

$$\mathbf{C}_+ = \{(x_0, x^+, x^-) \in E \mid \|x^-\| \le \|x^+\|/10\}$$

and

$$\mathbf{C}_- = \{(x_0, x^+, x^-) \in E \mid \|x^+\| \le \|x^-\|/10\}.$$

Definition 2.1. For $\lambda > 1$ and $\Lambda > 1$, let $\mathcal{H}(\lambda, \Lambda)$ be the set of C^r diffeomorphisms $G : V' \to V := G(V')$ on E satisfying the conditions

(H0) V' and V are open subsets in the unit disk $\mathbb{D} \subset E$,

(H1) $G^*(\alpha_0) = \alpha_0$ on V', and $G_*(v_0) = v_0$ on V, (H2) $DG_z(E \setminus \mathbf{C}_+) \subset \mathbf{C}_-$ and $(DG_z)^{-1}(E \setminus \mathbf{C}_-) \subset \mathbf{C}_+$ for any $z \in V'$,

(H3) $\|\pi_{+,-}(DG_z(v))\| \geq 2^\lambda \|\pi_{+,-}(v)\|$ for any $z \in V'$ and $v \in E \setminus \mathbf{C}_+$,
$\|\pi_{+,-}((DG_z)^{-1}(v))\| \geq 2^\lambda \|\pi_{+,-}(v)\|$ for any $z \in V'$ and $v \in E \setminus \mathbf{C}_-$,

(H4) $\det(DG_z|_Y) \geq 2^\Lambda$ for any $(d+1)$-dim subspaces $Y \subset \mathbf{C}_-$, and
$\det((DG_z)^{-1}|_{Y'}) \geq 2^\Lambda$ for any $(d+1)$-dim subspaces $Y' \subset \mathbf{C}_+$,

where $\det(A|_Y)$ is the expansion factor of the linear map $A : Y \to A(Y)$ with respect to the standard volumes on Y and $A(Y)$. Let $\mathcal{H}$ be the union of $\mathcal{H}(\lambda, \Lambda)$ for all $\lambda > 0$ and $\Lambda > 0$.

The following is a slight modification of the Darboux theorem.

Proposition 2.2. *There exists a finite system of Darboux charts on M,*

$$\kappa_a : U_a \to V_a := \kappa_a(U_a) \subset \mathbb{D} \subset E \quad \text{for } a \in A,$$

and a constant $c_0 > 0$ such that, if t is sufficiently large and if

$$V(a, b; t) := \kappa_a(U_a \cap F^{-t}(U_b)) \neq \emptyset \quad \text{for some } a, b \in A,$$

the induced diffeomorphism on the charts,

$$F^t_{ab} := \kappa_b \circ F^t \circ \kappa_a^{-1} : V(a, b; t) \to F^t_{ab}(V(a, b; t)) \subset V_b,$$

belongs to the class $\mathcal{H}(\lambda_0 t - c_0, \Lambda_0 t - c_0)$ defined above.

Proof. By compactness of M, it is enough to show, for each $z \in M$, that there exists a Darboux chart $\kappa : U \to V$ on a neighborhood U of z so that $\kappa(z) = 0$, $D\kappa_z(E^s(z)) = E_+$ and $D\kappa_z(E^u(z)) = E_-$. By Darboux theorem, there exists a Darboux chart $\kappa' : U' \to V'$ on a neighborhood U' of z so that $\kappa'(z) = 0$. For $E'_+ := D\kappa'_z(E^s(z))$ and $E'_- := D\kappa'_z(E^u(z))$, we have $E'_+ \oplus E'_- = D\kappa'_z(\mathrm{null}(\alpha_0(0))) = E_+ \oplus E_-$. Since $d\alpha$ is preserved by the flow F^t, we see $d\alpha|_{E^s} = d\alpha|_{E^u} = 0$ and therefore $d\alpha_0|_{E'_+} = d\alpha_0|_{E'_-} = 0$. So we can find a linear map $L : E_+ \oplus E_- \to E_+ \oplus E_-$ which preserves the symplectic form $d\alpha_0(0)|_{E_+ \oplus E_-}$ and staisfies $L(E'_+) = E_+$ and $L(E'_-) = E_-$. Define $L' : E \to E$ by $L'(x_0, x^+, x^-) = (x_0, L(x^+, x^-))$. Then it is easy to check that L' preserves the contact form α_0 and that the composition $\kappa := L' \circ \kappa'$ is a chart with the required properties. □

Henceforth we fix a finite system of Darboux charts $\kappa_a : U_a \to V_a$, $a \in A$, with the property in Proposition 2.2.

3. Transfer operators on local charts

In this section, we reduce Theorem 1.1 to the corresponding claim about transfer operators on the local charts. To state the claim, we prepare some definitions. For an open subset $V \subset E$, let $C^r(V)$ be the set of C^r functions whose supports are contained in V, and let $\mathscr{C}^r(V)$ be the subset of $g \in C^r(V)$ such that the differential $(v_0)^k g = \partial^k g / \partial x_0^k$ for arbitrarily large k exists and belongs to the class $C^r(V)$. We henceforth fix a large positive integer $r_* \geq 20(r+1)$ and set

$$\|g\|_* = \max_{0\leq k\leq r_*} \|\partial^k g/\partial x_0^k\|_{L^\infty} \quad \text{for } g \in \mathscr{C}^r(V).$$

For a C^r diffeomorphism $G : V' \to V$ in $\mathcal{H}$ and a function $g \in \mathscr{C}^r(V')$, we consider the transfer operator $\mathcal{L}(G,g) : C^r(V) \to C^r(V')$ defined by

$$\mathcal{L}(G,g)u(x) = \begin{cases} g(x)\cdot u(G(x)), & \text{for } x \in V'; \\ 0, & \text{otherwise.} \end{cases}$$

The Sobolev space $W^s(\mathbb{D})$ on the unit disk $\mathbb{D} \subset E$ is the completion of the space $C^\infty(\mathbb{D})$ with respect to the norm $\|u\|_{W^s} = \|(1+|\xi|^2)^{s/2}\cdot \mathbb{F}u(\xi)\|_{L^2}$, where $\mathbb{F} : L^2(E) \to L^2(E^*)$ is the Fourier transform.

Remark 3.1. The Sobolev space $W^s(M)$ for $|s| \leq r$ on M is defined from copies of $W^s(\mathbb{D})$ on the local charts by an obvious manner using a partition of unity. Clearly we have $C^r(M) \subset C^s(M) \subset W^s(M)$ for $0 \leq s \leq r$.

We will construct Hilbert spaces $\mathscr{B}^\beta_\nu$ for $\beta > 0$ and $\nu \geq 2d+2$, which satisfy $W^s(\mathbb{D}) \subset \mathscr{B}^\beta_\nu \subset W^{-s}(\mathbb{D})$ for $s > \beta$ and prove the following claims:

Theorem 3.2. *There exist positive constants λ_* and Λ_* so that the operator $\mathcal{L}(G,g)$ for any $G : V' \to V$ in $\mathcal{H}(\lambda_*, \Lambda_*)$ and $g \in \mathscr{C}^r(V')$ extends to a bounded operator $\mathcal{L}(G,g) : \mathscr{B}^\beta_\nu \to \mathscr{B}^\beta_{\nu'}$ for any $0 < \beta < (r-1)/2$ and $\nu, \nu' \geq 2\beta + 2d + 2$. Further, for any $\epsilon > 0$ and $0 < \beta < (r-1)/2$, there exist constants $\nu_* \geq 2\beta+2d+2$, $C_* > 0$ and a family of norms $\|\cdot\|^{(\lambda)}$ on $\mathscr{B}^\beta_{\nu_*}$ for $\lambda > 0$, which are all equivalent to the standard norm on $\mathscr{B}^\beta_{\nu_*}$, such that, if $G : V' \to V$ belongs to $\mathcal{H}(\lambda, \Lambda)$ for $\lambda \geq \lambda_*$ and $\Lambda \geq \Lambda_*$ with $\Lambda \geq d\lambda$ and if $g \in \mathscr{C}^r(V')$, there exists a compact operator $\mathcal{K}(G,g) : \mathscr{B}^\beta_{\nu_*} \to \mathscr{B}^\beta_{\nu_*}$ such that the operator norm of $\mathcal{L}(G,g) - \mathcal{K}(G,g) : \mathscr{B}^\beta_{\nu_*} \to \mathscr{B}^\beta_{\nu_*}$ with respect to the norm $\|\cdot\|^{(\lambda)}$ is bounded by $C_*\|g\|_* 2^{-(1-\epsilon)\min\{\Lambda/2, \beta\lambda\}}$.*

We show that Theorem 1.1 follows from Theorem 3.2. Take C^r functions $\rho_a : V_a \to [0,1]$ and $\widetilde{\rho}_a : V_a \to [0,1]$ for $a \in A$ so that the family $\{\rho_a \circ \kappa_a\}_{a\in A}$ is a C^r partition of unity on M and that $\widetilde{\rho}_a \equiv 1$ on $\operatorname{supp}\rho_a$ and $\operatorname{supp}\widetilde{\rho}_a \Subset V_a$. We may and do suppose that ρ_a and $\widetilde{\rho}_a$ belong to the class $\mathscr{C}^r(V_a)$, applying an appropriate C^∞ mollifier along the coordinate x_0 simultaneously(3).

For $a, b \in A$, we define the transfer operator $\mathcal{L}^t_{ab} : C^r(V_b) \to C^r(V_a)$ by

$$\mathcal{L}^t_{ab}u(x) = \begin{cases} g^t_{ab}(x)\cdot u(F^t_{ab}(x)), & \text{if } x \in V(a,b;t); \\ 0, & \text{otherwise} \end{cases}$$

(3)This is possible because the coordinate transformations $\kappa_a \circ \kappa_b^{-1}$ preserve the vector field v_0.

where $g^t_{ab}(x) = \rho_a(x) \cdot \widetilde{\rho}_b(F^t_{ab}(x))$ belongs to $\mathscr{C}^r(V_a)$. Then we consider the matrix of operators

$$\mathbf{L}^t : \oplus_{a\in A} C^r(V_a) \to \oplus_{a\in A} C^r(V_a), \quad \mathbf{L}^t((u_a)_{a\in A}) = \left(\sum_{b\in A} \mathcal{L}^t_{ab}(u_b)\right)_{a\in A}.$$

Let $\iota : C^r(M) \to \oplus_{a\in A} C^r(V_a)$ be the injection defined by

$$\iota(u) = (\rho_a \cdot (u \circ \kappa_a^{-1}))_{a\in A}.$$

By the definition, we have the commutative diagram

$$\begin{array}{ccc} \oplus_{a\in A} C^r(V_a) & \xrightarrow[\mathbf{L}^t]{} & \oplus_{a\in A} C^r(V_a) \\ \uparrow \iota & & \uparrow \iota \\ C^r(M) & \xrightarrow[\mathcal{L}^t]{} & C^r(M) \end{array}$$

Let B^β_ν be the completion of $C^r(M)$ with respect to the pull-back of the product norm on $\oplus_{a\in A} \mathscr{B}^\beta_\nu \supset \oplus_{a\in A} C^r(V_a)$ by the injection ι, so that the injection ι extends to the isometric embedding $\iota : B^\beta_\nu \to \oplus_{a\in A} \mathscr{B}^\beta_\nu$ and that $W^s(M) \subset B^\beta_\nu \subset W^{-s}(M)$ for $s > \beta$.

Let c_0 be the constant in Proposition 2.2, and λ_* and Λ_* those in the former statement of Theorem 3.2. Take $t_0 > 0$ so large that $\lambda_0 t_0 - c_0 \geq \lambda_*$ and $\Lambda_0 t_0 - c_0 \geq \Lambda_*$. Applying the former statement of Theorem 3.2 to each $\mathcal{L}^t_{ab}$, we see that the commutative diagram above extends to

$$\begin{array}{ccc} \oplus_{a\in A} \mathscr{B}^\beta_\nu & \xrightarrow[\mathbf{L}^t]{} & \oplus_{a\in A} \mathscr{B}^\beta_{\nu'} \\ \uparrow \iota & & \uparrow \iota \\ B^\beta_\nu & \xrightarrow[\mathcal{L}^t]{} & B^\beta_{\nu'} \end{array}$$

for any $t \geq t_0$, provided that $0 < \beta < (r-1)/2$ and $\nu, \nu' \geq 2\beta + 2d + 2$.

Suppose that $\epsilon > 0$ and $0 < \beta < (r-1)/2$ are given arbitrarily and let ν_*, C_* and $\|\cdot\|^{(\lambda)}$ be those in the latter statement of Theorem 3.2. Recall that the essential spectral radius of an operator on a Banach space coincides with the infimum of the spectral radii of its purturbations by compact operators. (See [23].) Hence, applying the latter statement of Theorem 3.2 to each $\mathcal{L}^t_{ab}$, we see that the essential spectral radius of $\mathbf{L}^t : \oplus_{a\in A} \mathscr{B}^\beta_{\nu_*} \to \oplus_{a\in A} \mathscr{B}^\beta_{\nu_*}$ is bounded by

$$C_* \cdot \#A \cdot \left(\max_{a,b\in A} \|g^t_{ab}\|_* \right) \cdot 2^{-(1-\epsilon)\min\{(\Lambda_0 t - c_0)/2, \beta(\lambda_0 t - c_0)\}}$$

and so is that of $\mathcal{L}^t : B^\beta_{\nu_*} \to B^\beta_{\nu_*}$ from the commutative diagram above. Note that the term $\max_{a,b\in A} \|g^t_{ab}\|_*$ is bounded by a constant independent of t, because F^t_{ab} preserves the standard vector field v_0. From the multiplicative property of essential spectral radius, the essential spectral radius of $\mathcal{L}^t$:

$B^{\beta}_{\nu_*} \to B^{\beta}_{\nu_*}$ is bounded by $2^{-(1-\epsilon)\min\{\Lambda_0 t/2, \beta\lambda_0 t\}}$. Fix some $\nu \geq 2\beta + 2d + 2$ arbitrarily and decompose $\mathcal{L}^t : B^{\beta}_{\nu} \to B^{\beta}_{\nu}$ for $t > 3t_0$ as

$$B^{\beta}_{\nu} \xrightarrow{\mathcal{L}^{t_0}} B^{\beta}_{\nu_*} \xrightarrow{\mathcal{L}^{t-2t_0}} B^{\beta}_{\nu_*} \xrightarrow{\mathcal{L}^{t_0}} B^{\beta}_{\nu}$$

Letting $t \to \infty$ and using the basic properties of essential spectral radius mentioned above, we see that the essential spectral radius of $\mathcal{L}^t : B^{\beta}_{\nu} \to B^{\beta}_{\nu}$ is bounded by that of $\mathcal{L}^t : B^{\beta}_{\nu_*} \to B^{\beta}_{\nu_*}$ and hence by $2^{-(1-\epsilon)\min\{\Lambda_0 t/2, \beta\lambda_0 t\}}$. Since $\epsilon > 0$ is arbitrary, we obtain the main theorem, setting $B^{\beta} = B^{\beta}_{\nu}$.

4. A local geometric property of the diffeomorphisms in $\mathcal{H}$

In this section, we give a local geometric property of the diffeomorphisms in $\mathcal{H}$. Let $G : V' \to V = G(V')$ be a C^r diffeomorphism satisfying the conditions (H0) and (H1) in the definition of $\mathcal{H}(\lambda, \Lambda)$. Take a small disk $D \subset V'$ and set $D' = \pi_{+,-}(D)$. Since G preserves the standard vector field v_0, there exist a C^r function $G_0 : D' \to \mathbb{R}$ and a C^r diffeomorphism

$$G_{+,-} : D' \to G_{+,-}(D') \subset \mathbb{R}^{2d}, \quad G_{+,-}(x^+, x^-) = (G_+(x^+, x^-), G_-(x^+, x^-)),$$

such that

$$G(x_0, x^+, x^-) = (x_0 + G_0(x^+, x^-), G_+(x^+, x^-), G_-(x^+, x^-)) \quad \text{on } D.$$

Lemma 4.1. *If $G(0) = 0 \in D$ in addition, we have $DG_0(0) = D^2 G_0(0) = 0$.*

Proof. Comparing the coefficients of dx^+ and dx^- in $G^*(\alpha_0) = \alpha_0$, we get

$$\frac{\partial G_0}{\partial x^+} = -G_- \cdot \frac{\partial G_+}{\partial x^+} + G_+ \cdot \frac{\partial G_-}{\partial x^+} + x^-$$

and

$$\frac{\partial G_0}{\partial x^-} = -G_- \cdot \frac{\partial G_+}{\partial x^-} + G_+ \cdot \frac{\partial G_-}{\partial x^-} - x^+.$$

This implies $\partial G_0/\partial x^+(0) = \partial G_0/\partial x^-(0) = 0$. Differentiating the both sides with respect to x^+ and x^- and using the assumption $G(0) = 0$, we also obtain $\partial^2 G_0/\partial x^+ \partial x^+(0) = \partial^2 G_0/\partial x^+ \partial x^-(0) = \partial^2 G_0/\partial x^- \partial x^-(0) = 0$. □

For $y = (y_0, y^+, y^-) \in E$, the affine bijection $\Phi_y : E \to E$ defined by

$$\Phi_y(x_0, x^+, x^-) = (y_0 + x_0 - (y^- \cdot x^+) + (y^+ \cdot x^-),\ y^+ + x^+,\ y^- + x^-) \tag{1}$$

moves the origin 0 to y, preserving the contact form α_0 and the vector field v_0. So the assumption $G(0) = 0$ in Lemma 4.1 is not essential.

Corollary 4.2. *For any diffeomorphism $G : V' \to V$ in $\mathcal{H}$ and any compact subset K of V', there exists a constant $C > 0$ such that, if $y, y' \in K$ and if $\xi \in E^*$ is written in the form $\xi = \xi_0 \cdot \alpha_0(G(y)) + \xi_{+,-}$ with $\xi_0 = \pi_0^*(\xi)$ and $\xi_{+,-} \in E^*_+ \oplus E^*_-$, we have*

$$\|DG^*_{y'}(\xi) - DG^*_y(\xi)\| \leq C \cdot (|\xi_0| \cdot \|y' - y\|^2 + \|\xi_{+,-}\| \cdot \|y' - y\|).$$

Proof. Changing coordinates by the affine bijections Φ_y and $\Phi_{G(y)}$, we may suppose $y = G(y) = 0$. Then the claim follows from Lemm 4.1. □

5. Partitions of Unity

In this section, we construct a partition of unity $\{p_\gamma\}_{\gamma\in\Gamma}$ on the cotangent bundle $T^*_{\mathbb{D}}E = \mathbb{D}\times E^*$ over the unit disk $\mathbb{D}\subset E$. This will be used in the definition of the Hilbert spaces $\mathscr{B}^\beta_\nu$ in the next section.

5.1. **Partitions of unity on E.** Take a C^∞ function $\chi:\mathbb{R}\to[0,1]$ so that

$$\chi(s)=\begin{cases}1, & \text{if } s\le 4/3;\\ 0, & \text{if } s\ge 5/3,\end{cases}$$

and define a C^∞ function $\rho:\mathbb{R}\to[0,1]$ by $\rho(s)=\chi(s+1)-\chi(s+2)$. Then ρ is supported on the interval $[-2/3,2/3]$ and the family of functions $\{\rho(\cdot+k)\,;\,k\in\mathbb{Z}\}$ is a C^∞ partition of unity on $\mathbb{R}$.

For integers $n\ge 0$ and k, we define the C^∞ function $\rho_{n,k}:\mathbb{R}\to[0,1]$ by

$$\rho_{n,k}(s)=\rho(2^{n/2}s-k).$$

Similarly, for $n\ge 0$ and $\mathbf{k}=(k_0,k_1^+,\cdots,k_d^+,k_1^-,\cdots,k_d^-)\in\mathbb{Z}^{2d+1}$, we define the C^∞ function $\rho_{n,\mathbf{k}}:E\to[0,1]$ by

$$\rho_{n,\mathbf{k}}(x)=\rho(2^{n/2}x_0-k_0)\prod_{\sigma=\pm}\prod_{i=1}^{d}\rho(2^{n/2}x_i^\sigma-k_i^\sigma).$$

Then, for each $n\ge 0$, the families of functions $\{\rho_{n,k}(s)\mid k\in\mathbb{Z}\}$ and $\{\rho_{n,\mathbf{k}}(s)\mid \mathbf{k}\in\mathbb{Z}^{2d+1}\}$ are C^∞ partitions of unity on $\mathbb{R}$ and E respectively. The support of $\rho_{n,k}$ is contained in $[2^{-n/2}(k-1),2^{-n/2}(k+1)]$ and that of $\rho_{n,\mathbf{k}}$ is contained in the cube

$$Z(n,\mathbf{k})=[2^{-n/2}(k_0-1),2^{-n/2}(k_0+1)]\times\prod_{\sigma=\pm}\prod_{i=1}^{d}[2^{-n/2}(k_i^\sigma-1),2^{-n/2}(k_i^\sigma+1)],$$

whose center is at the point

$$z(n,\mathbf{k}):=2^{-n/2}\mathbf{k}=2^{-n/2}(k_0,k_1^+,\dots,k_d^+,k_1^-,\dots,k_d^-).$$

5.2. **Partitions of unity on E^*.** We next introduce a few partitions of unity on the dual space E^*. For $n\ge 0$, we consider the C^∞ functions

$$\chi_n:\mathbb{R}\to[0,1],\qquad \chi_n(s)=\begin{cases}\chi(2^{-n}|s|)-\chi(2^{-n+1}|s|), & \text{if } n\ge 1;\\ \chi(|s|), & \text{if } n=0.\end{cases}$$

and

$$\widetilde{\chi}_n:\mathbb{R}\to[0,1],\qquad \widetilde{\chi}_n(s)=\begin{cases}\chi_{n-1}(s)+\chi_n(s)+\chi_{n+1}(s), & \text{if } n\ge 1;\\ \chi_0(s)+\chi_1(s), & \text{if } n=0.\end{cases}$$

The family of functions χ_n for $n\ge 0$ is a C^∞ partition of unity on $\mathbb{R}$ and we have $\widetilde{\chi}_n\equiv 1$ on $\operatorname{supp}\chi_n$ for each $n\ge 0$.

For $n \geq 0$ and $k \in \mathbb{Z}$, we consider the C^∞ functions

$$\chi_{n,k} : E^* \to [0,1], \quad \chi_{n,k}(\xi) = \rho_{n,k}(\xi_0) \cdot \chi_n(\xi_0) \quad \text{where } \xi_0 = \pi_0^*(\xi),$$

and

$$\widetilde{\chi}_{n,k} : E^* \to [0,1], \quad \widetilde{\chi}_{n,k}(\xi) = \rho_{n,k-1}(\xi_0) + \rho_{n,k}(\xi_0) + \rho_{n,k+1}(\xi_0).$$

Then the family of functions $\{\chi_{n,k} \mid n \geq 0, k \in \mathbb{Z}\}$ is a C^∞ partition of unity on E and we have $\widetilde{\chi}_{n,k} \equiv 1$ on $\operatorname{supp} \chi_{n,k}$ for each n and k.

Remark 5.1. We will ignore the functions $\chi_{n,k}$ that vanish everywhere. Thus, for given $n \geq 0$, we consider the functions $\chi_{n,k}$ only for finitely many k's.

Let S^* be the unit sphere in $E_+^* \oplus E_-^*$. We henceforth fix C^∞ functions $\varphi_\sigma : S^* \to [0,1]$ and $\widetilde{\varphi}_\sigma : S^* \to [0,1]$ for $\sigma \in \{+,-\}$ such that

(i) $\varphi_\sigma \equiv 1$ on a neighborhood $S^* \cap C_\sigma^*(4/10)$ for $\sigma = \pm$,
(ii) $\varphi_+(\xi) + \varphi_-(\xi) = 1$ for all $\xi \in S^*$,
(iii) $\widetilde{\varphi}_\sigma \equiv 1$ on $C_\sigma^*(6/10)$ and $\operatorname{supp} \widetilde{\varphi}_\sigma \subset C_\sigma^*(7/10) \cap S^*$ for $\sigma = \pm$.

For an integer m, let $\psi_m : E_+^* \oplus E_-^* \to [0,1]$ and $\widetilde{\psi}_m : E_+^* \oplus E_-^* \to [0,1]$ be C^∞ functions defined respectively by

$$\psi_m(\xi) = \begin{cases} \chi_m(\|\xi\|) \cdot \varphi_+(\xi/\|\xi\|), & \text{if } m \geq 1; \\ \chi_0(\|\xi\|), & \text{if } m = 0; \\ \chi_{|m|}(\|\xi\|) \cdot \varphi_-(\xi/\|\xi\|), & \text{if } m \leq -1 \end{cases}$$

and

$$\widetilde{\psi}_m(\xi) = \begin{cases} \widetilde{\chi}_m(\|\xi\|) \cdot \widetilde{\varphi}_+(\xi/\|\xi\|), & \text{if } m \geq 1; \\ \widetilde{\chi}_0(\|\xi\|), & \text{if } m = 0; \\ \widetilde{\chi}_{|m|}(\|\xi\|) \cdot \widetilde{\varphi}_-(\xi/\|\xi\|), & \text{if } m \leq 1. \end{cases}$$

Then $\{\psi_m\}_{m \in \mathbb{Z}}$ is a C^∞ partition of unity on the subspace $E_+^* \oplus E_-^*$ and we have $\widetilde{\psi}_m \equiv 1$ on $\operatorname{supp} \psi_m$. Next we define C^∞ functions $\psi_{n,k,m} : E^* \to [0,1]$ and $\widetilde{\psi}_{n,k,m} : E^* \to [0,1]$ for $n \geq 0$ and $k, m \in \mathbb{Z}$ respectively by

$$\psi_{n,k,m}(\xi) = \chi_{n,k}(\xi) \cdot \psi_m(2^{-n/2}\xi^+, 2^{-n/2}\xi^-)$$

and

$$\widetilde{\psi}_{n,k,m}(\xi) = \widetilde{\chi}_{n,k}(\xi) \cdot \widetilde{\psi}_m(2^{-n/2}\xi^+, 2^{-n/2}\xi^-).$$

Then the family $\{\psi_{n,k,m} \mid n \geq 0, m, k \in \mathbb{Z}\}$ is a C^∞ partition of unity on E^* and we have $\widetilde{\psi}_{n,k,m} \equiv 1$ on $\operatorname{supp} \psi_{n,k,m}$.

5.3. **Partitions of unity on $T_{\mathbb{D}}^* E = \mathbb{D} \times E^*$.** Recalling Remark 5.1, we set

$$\mathcal{N} = \{(n,k) \in \mathbb{Z}_+ \oplus \mathbb{Z} \mid \chi_{n,k} \text{ does not vanish.}\},$$

and then let

$$\Gamma = \left\{(n,k,m,\mathbf{k}) \in \mathcal{N} \oplus \mathbb{Z} \oplus \mathbb{Z}^{2d+2} \mid \operatorname{supp} \rho_{n,\mathbf{k}} \cap \mathbb{D} \neq \emptyset\right\}.$$

To refer the components of $\gamma = (n,k,m,\mathbf{k}) \in \Gamma$, we set

$$n(\gamma) = n, \quad k(\gamma) = k, \quad m(\gamma) = m \quad \text{and} \quad \mathbf{k}(\gamma) = \mathbf{k}.$$

And we put $\rho_\gamma = \rho_{n(\gamma),\mathbf{k}(\gamma)}$, $Z(\gamma) = Z(n(\gamma),\mathbf{k}(\gamma))$ and $z(\gamma) = z(n(\gamma),\mathbf{k}(\gamma))$. Recall the diffeomorphism $\Phi_y : E \to E$ defined for $y \in E$ by (1). For each $\gamma \in \Gamma$, we consider the linear map

$$\Phi_\gamma = ((D\Phi_{z(\gamma)})_0)^* : T_{z(\gamma)}E^* \to T_0E^*,$$

which satisfies $\Phi_\gamma(\alpha_0(z(\gamma))) = \alpha_0(0)$ and $\Phi_\gamma|_{E_+^*\oplus E_-^*} = id$. We then define the C^∞ functions $\psi_\gamma : E^* \to [0,1]$ and $\widetilde{\psi}_\gamma : E^* \to [0,1]$ by

$$\psi_\gamma = \psi_{n(\gamma),k(\gamma),m(\gamma)} \circ \Phi_\gamma \quad \text{and} \quad \widetilde{\psi}_\gamma = \widetilde{\psi}_{n(\gamma),k(\gamma),m(\gamma)} \circ \Phi_\gamma.$$

Finally we define the family of C^∞ functions $p_\gamma : T^*E \to [0,1]$ for $\gamma \in \Gamma$ by

$$p_\gamma(x,\xi) = \rho_\gamma(x)\cdot\psi_\gamma(\xi) \quad \text{for } (x,\xi) \in T^*E = E \times E^*.$$

This family is a C^∞ partition of unity on $T^*_{\mathbb{D}}E = \mathbb{D}\times E^*$. In fact, for given $(n,k) \in \mathcal{N}$ and $\mathbf{k} \in \mathbb{Z}^{2d+1}$, we have

$$\sum_{\gamma: n(\gamma)=n; k(\gamma)=k; \mathbf{k}(\gamma)=\mathbf{k}} p_\gamma(x,\xi) = \rho_{n,\mathbf{k}}(x)\cdot\chi_{n,k}(\xi) \quad \text{for } (x,\xi) \in T^*E$$

and hence

$$\sum_{\gamma\in\Gamma} p_\gamma(x,\xi) \equiv 1 \quad \text{for } (x,\xi) \in T^*_{\mathbb{D}}E = \mathbb{D}\times E^*.$$

5.4. **Boundedness of the family ψ_γ and $\widetilde{\psi}_\gamma$ up to scaling.** For integers $n \geq 0$ and m, let $J_{n,m} : E^* \to E^*$ be the linear map defined by

$$J_{n,m}(\xi_0,\xi^+,\xi^-) = (2^{n/2}\xi_0, 2^{n/2+|m|}\xi^+, 2^{n/2+|m|}\xi^-).$$

For $\gamma \in \Gamma$, let $A_\gamma : E^* \to E^*$ be the translation defined by

$$A_\gamma(\xi) = \xi + k(\gamma)\cdot 2^{n(\gamma)/2}\cdot\alpha_0(z(\gamma)).$$

Since $z(\gamma)$ for $\gamma \in \Gamma$ are bounded, it is not difficult to see

Lemma 5.2. *$\psi_\gamma \circ A_\gamma \circ J_{n(\gamma),m(\gamma)}$ and $\widetilde{\psi}_\gamma \circ A_\gamma \circ J_{n(\gamma),m(\gamma)}$ for $\gamma \in \Gamma$ are all supported in a bounded subset in E and their C^s norms are uniformly bounded for every $s \geq 0$.*

For $n \in \mathbb{Z}_+$, $m \in \mathbb{Z}$ and $\mu > 0$, we define the function $b^\mu_{n,m} : E \to \mathbb{R}_+$ by

$$b^\mu_{n,m}(x) = |\det J_{n,m}|\cdot\langle J_{n,m}(x)\rangle^{-\mu}, \tag{2}$$

where (and henceforth) we set

$$\langle y\rangle = (1+\|y\|^2)^{1/2}.$$

For brevity, we set $b^\mu_\gamma = b^\mu_{n(\gamma),m(\gamma)}$ for $\gamma \in \Gamma$. Then the last lemma implies

Corollary 5.3. *For each $\mu > 0$, there exists a constant $C_* > 0$ such that $|\mathbb{F}^{-1}\psi_\gamma(x)| \leq C_*\cdot b^\mu_\gamma(x)$ and $|\mathbb{F}^{-1}\widetilde{\psi}_\gamma(x)| \leq C_*\cdot b^\mu_\gamma(x)$ for all $x \in E$ and $\gamma \in \Gamma$.*

6. The Hilbert spaces $\mathscr{B}^\beta_\nu$

In this section, we define the Hilbert spaces $\mathscr{B}^\beta_\nu$ in Theorem 3.2.

6.1. **Decomposition of functions using pseudodifferential operators.** For a C^∞ function $p : T^*E \to \mathbb{R}$ on the cotangent bundle $T^*E = E \times E^*$ with compact support, the adjoint of the pseudodifferential operator $p_\gamma(x, D)$ with symbol p_γ is the operator $p(x, D)^* : L^2(E) \to L^2(E)$ given by

$$p(x, D)^* u(x) = (2\pi)^{-(2d+1)} \int e^{i\langle \xi, x-y\rangle} p(y, \xi) u(y) dy d\xi.$$

For a C^∞ function $\psi : E^* \to \mathbb{R}$ with compact support, we consider the operator $\psi(D) : L^2(E) \to L^2(E)$ defined similarly by

$$\begin{aligned}\psi(D)u(x) &= (2\pi)^{-(2d+1)} \int e^{i\langle \xi, x-y\rangle} \psi(\xi) u(y) dy d\xi \\ &= \mathbb{F}^{-1}(\psi \cdot \mathbb{F}u)(x) = (\mathbb{F}^{-1}\psi) * u(x).\end{aligned}$$

Remark 6.1. Each of the notations $p(x, D)^*$ and $\psi(D)$ above should be read as a single symbol and the letter x and D have no meaning as variable. We refer [16, 29] for the definition of pseudodifferential operator.

For $u \in L^2(\mathbb{D})$ and $\gamma \in \Gamma$, we set

$$u_\gamma := p_\gamma(x, D)^* u = \psi_\gamma(D)(\rho_\gamma \cdot u) = (\mathbb{F}^{-1}\psi_\gamma) * (\rho_\gamma \cdot u).$$

Then we have $u = \sum_{\gamma\in\Gamma} u_\gamma$ in $L^2(\mathbb{D})$, because $\{p_\gamma\}_{\gamma\in\Gamma}$ is a partition of unity on $T^*_{\mathbb{D}}E$. Note that each u_γ is localized near $\operatorname{supp} \rho_\gamma$ by Corollary 5.3 and its Fourier transform is supported in $\operatorname{supp} \psi_\gamma$ by definition.

6.2. **The definition of the Hilbert space $\mathscr{B}^\beta$.** For $\beta > 0$ and $\nu \geq 2d+2$, we set

$$\|u\|_{\beta,\nu} = \left(\sum_{\gamma\in\Gamma} 2^{2\beta m(\gamma)} \|d^\nu_\gamma \cdot u_\gamma\|^2_{L^2} \right) \qquad \text{for } u \in C^\infty(\mathbb{D}),$$

where $d^\nu_\gamma : E \to \mathbb{R}$ is the function defined by

$$d^\nu_\gamma(x) = \langle 2^{n(\gamma)/2}(x - z(\gamma))\rangle^\nu = \left(1 + 2^{n(\gamma)}\|x - z(\gamma)\|^2\right)^{\nu/2}.$$

This is a norm on $C^\infty(\mathbb{D})$ associated to a unique inner product $(\cdot, \cdot)_{\beta,\nu}$. Further we have

Lemma 6.2. *For $0 < \beta < s$ and $\nu \geq 2d + 2$, there exists a constant $C > 0$ such that $(1/C)\|u\|_{W^{-s}} \leq \|u\|_{\beta,\nu} \leq C\|u\|_{W^s}$ for all $u \in C^\infty(\mathbb{D})$.*

We give the proof of this lemma in the appendix at the end of this paper, one because it requires some estimates that will be given in the following sections. Now we define the Hilbert space $\mathscr{B}^\beta_\nu$ as follows

Definition 6.3. For $0 < \beta < (r-1)/2$ and $\nu \geq 2d+2$, the Hilbert space $\mathscr{B}^\beta_\nu$ is the completion of the space $C^\infty(\mathbb{D})$ with respect to the norm $\|\cdot\|_{\beta,\nu}$, equipped with the extension of the inner product $(\cdot, \cdot)_{\beta,\nu}$.

Then the first claim of Theorem 3.2 follows immediately from Lemma 6.2.

7. The auxiliary operator $\mathcal{M}(G,g)$

In this section, we introduce the operator $\mathcal{M}(G,g) : \mathbf{B}^\beta_\nu \to \mathbf{B}^\beta_{\nu'}$ between Hilbert spaces. This is an extension of the operator $\mathcal{L}(G,g)$ in the sense that there exists an isometric embedding $\iota : \mathscr{B}^\beta_\nu \to \mathbf{B}^\beta_\nu$ and that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{B}^\beta_\nu & \xrightarrow{\mathcal{M}(G,g)} & \mathbf{B}^\beta_{\nu'} \\ \uparrow \iota & & \uparrow \iota \\ \mathscr{B}^\beta_\nu & \xrightarrow{\mathcal{L}(G,g)} & \mathscr{B}^\beta_{\nu'} \end{array} \tag{3}$$

7.1. **The definition of the operator $\mathcal{M}$.** For $\beta > 0$ and $\nu \geq 2d+2$, we consider the Hilbert space $\mathbf{B}^\beta_\nu \subset (L^2(E))^\Gamma$ defined by

$$\mathbf{B}^\beta_\nu = \left\{ \mathbf{u} = (u_\gamma)_{\gamma\in\Gamma} \;\middle|\; \widetilde{\psi}_\gamma(D)u_\gamma = u_\gamma,\ \sum_{\gamma\in\Gamma} 2^{2\beta m(\gamma)} \|d^\nu_\gamma \cdot u_\gamma\|^2_{L^2} < \infty \right\}$$

and equipped with the norm $\|\mathbf{u}\|_{\beta,\nu} = \sum_{\gamma\in\Gamma} 2^{2\beta m(\gamma)} \|d^\nu_\gamma \cdot u_\gamma\|^2_{L^2}$. Then the injection $\iota : \mathscr{B}^\beta_\nu \to \mathbf{B}^\beta_\nu$, $\iota(u) = (p_\gamma(x,D)^* u)_{\gamma\in\Gamma}$, is an isometric embedding.

Suppose that $v = \mathcal{L}(G,g)u$ for $u \in L^2(\mathbb{D})$ and set $u_\gamma = p_\gamma(x,D)^*u$ and $v_\gamma = p_\gamma(x,D)^*v$ for $\gamma \in \Gamma$. Then we have

$$v_{\gamma'} = \sum_{\gamma\in\Gamma} \mathcal{L}_{\gamma\gamma'} u_\gamma, \tag{4}$$

where the operator $\mathcal{L}_{\gamma\gamma'} = \mathcal{L}_{\gamma\gamma'}(G,g) : L^2(E) \to L^2(E)$ is defined by

$$\mathcal{L}_{\gamma\gamma'} w = p_{\gamma'}(x,D)^*(\mathcal{L}(G,g)(\widetilde{\psi}_\gamma(D)w)). \tag{5}$$

Remark 7.1. Since $\widetilde{\psi}_\gamma(D)u_\gamma = u_\gamma$ in the setting above, the operation $\widetilde{\psi}_\gamma(D)$ in (5) is not necessary for (4) to hold. But this operation makes difference when we regard $\mathcal{L}_{\gamma\gamma'}$ as an operator on $L^2(E)$.

We define the operator $\mathcal{M}(G,g) : \mathbf{B}^\beta_\nu \to \mathbf{B}^\beta_{\nu'}$ formally by

$$\mathcal{M}(G,g)((u_\gamma)_{\gamma\in\Gamma}) = \left(\sum_{\gamma\in\Gamma} \mathcal{L}_{\gamma\gamma'}(u_\gamma) \right)_{\gamma'\in\Gamma}. \tag{6}$$

Then, by (4), the diagram (3) commutes in the formal level at least. In the following sections, we will prove

Theorem 7.2. *There exist constants $\lambda_* > 0$ and $\Lambda_* > 0$ such that, for $G : V' \to V$ in $\mathcal{H}(\lambda_*, \Lambda_*)$ and $g \in \mathscr{C}^r(V')$, the formal definition of $\mathcal{M}(G,g)$ gives a bounded operator $\mathcal{M}(G,g) : \mathbf{B}^\beta_\nu \to \mathbf{B}^\beta_{\nu'}$ for $0 < \beta < (r-1)/2$ and $\nu, \nu' \geq 2\beta + 2d + 2$, and makes the diagram (3) commutes.*

Further, for any $\epsilon > 0$ and $0 < \beta < (r-1)/2$, there exist $\nu_* \geq 2\beta+2d+2$, $C_* > 0$ and a family of norms $\|\cdot\|^{(\lambda)}$ on $\mathbf{B}^{\beta}_{\nu_*}$ for $\lambda > 0$, which are equivalent to the norm defined from the inner product on $\mathbf{B}^{\beta}_{\nu_*}$, such that, if $G : V' \to V$ belongs to $\mathcal{H}(\lambda, \Lambda)$ for $\lambda \geq \lambda_*$ and $\Lambda \geq \Lambda_*$ with $\Lambda \geq d\lambda$ and if $g \in \mathscr{C}^r(V')$, there exists a compact operator $\mathcal{K}(G, g) : \mathbf{B}^{\beta}_{\nu_*} \to \mathbf{B}^{\beta}_{\nu_*}$ such that the operator norm of $\mathcal{M}(G, g) - \mathcal{K}(G, g) : \mathbf{B}^{\beta}_{\nu_*} \to \mathbf{B}^{\beta}_{\nu_*}$ with respect to the norm $\|\cdot\|^{(\lambda)}$ is bounded by $C_* \cdot \|g\|_* \cdot 2^{-(1-\epsilon)\min\{\Lambda/2, \beta\lambda\}}$.

Since the operator ι in (3) is an isometric embedding, Theorem 3.2 follows from Theorem 7.2 immediately.

7.2. **The operator $\mathcal{L}_{\gamma\gamma'}$.** The operator $\mathcal{L}_{\gamma\gamma'} : L^2(E) \to L^2(E)$ defined in the last subsection can be regarded as an integral operator

$$\mathcal{L}_{\gamma\gamma'} u(x') = \int \kappa_{\gamma\gamma'}(x', x) u(x) dx$$

with the kernel

$$\begin{aligned} \kappa_{\gamma\gamma'}(x', x) &= \int \mathbb{F}^{-1}\psi_{\gamma'}(x' - y) \cdot \rho_{\gamma'}(y) \cdot g(y) \cdot \mathbb{F}^{-1}\tilde{\psi}_{\gamma}(G(y) - x) dy \\ &= (2\pi)^{-2(2d+1)} \int e^{i\langle \xi, x'-y\rangle + i\langle \eta, G(y)-x\rangle} \rho_{\gamma'}(y) g(y) \psi_{\gamma'}(\xi) \tilde{\psi}_{\gamma}(\eta) d\xi d\eta dy. \qquad (7) \end{aligned}$$

It follows from Corollary 5.3 that

Lemma 7.3. *For each $\mu > 0$, there exists a constant $C_* > 0$ such that*

$$|\kappa_{\gamma\gamma'}(x', x)| \leq C_* \cdot \|g\|_{L^\infty} \cdot \int_{Z(\gamma')} b^{\mu}_{\gamma'}(x' - y) \cdot b^{\mu}_{\gamma}(G(y) - x) dy$$

for $(x, x') \in E \times E$, $\gamma, \gamma' \in \Gamma$ and for $G : V' \to V$ in $\mathcal{H}$ and $g \in \mathscr{C}^r(V')$.

This uniform estimate is quite useful. But we can and need to improve this estimate in some cases. In the case where $DG^*_y(\operatorname{supp} \widetilde{\psi}_{\gamma})$ for $y \in \operatorname{supp} \rho_{\gamma'}$ are apart from $\operatorname{supp} \psi_{\gamma'}$, it is natural to expect that the operator norm of $\mathcal{L}_{\gamma\gamma'}$ is small. To justify this idea, we use the fact that the term $e^{i\langle \xi, x'-y\rangle + i\langle \eta, G(y)-x\rangle}$ in (7) oscillates fast in such case and therefore the integration with respect to the variable y in (7) can be regarded as an oscillatory integral.

Let us recall a technique in estimating oscillatory integrals. (See [17, §7.7] for more details.) Consider an integral of the form

$$\int h(x) e^{if(x)} dx \qquad (8)$$

where $h(x)$ is a continuous function supported on a compact subset in E and $f(x)$ a real-valued continuous function defined on a neighborhood of the support of h. Take a few vectors $v_1, v_2, \ldots, v_k$ in E and regard them as constant vector fields on E. Assume that the functions f and h are so smooth that $v_i f$, $v_i v_j f$ and $v_i h$ for $1 \leq i, j \leq k$ exist and are continuous on a neighborhood of the support of h and also that

$$v_1(f)^2 + v_2(f)^2 + \cdots + v_k(f)^2 \neq 0 \quad \text{on the support of } h.$$

Then we can apply integration by part to obtain

$$\int h(x)e^{if(x)}dx = \int Lh(x)e^{if(x)}dx$$

where

$$Lh = \sum_{j=1}^{k} v_j \left(\frac{i \cdot h \cdot v_j(f)}{\sum_{j=1}^{k} v_j(f)^2} \right).$$

This formula tells that if the term $e^{if(x)}$ oscillate fast in the directions spanned by the vectors $v_1, v_2, \cdots, v_k$, the term $Lh(x)$ is small and so is the oscillatory integral (8).

Assuming more smoothness of the functions f and h, we may repeat the operation above and obtain the formula

$$\int h(x)e^{if(x)}dx = \int L^{\ell}h(x)e^{if(x)}dx. \tag{9}$$

Basically we get better estimate if we exploit this formula for larger ℓ. This is the point where differentiablity of the flow get into our argument.

Below we give a simple estimate on the kernel $\kappa_{\gamma\gamma'}$ applying the formula (9). Though this estimate is still not enough for our argument, it is a good starting point. For integers n, k, n', k' such that $(n,k), (n',k') \in \mathcal{N}$, we set

$$\Delta(n,k,n',k') = \log_2^+ \left(2^{-n'/2} \cdot d(\operatorname{supp} \widetilde{\chi}_{n,k}, \operatorname{supp} \chi_{n',k'}) \right)$$

where $\log_2^+ t = \max\{0, \log_2 t\}$. Also we put

$$\widetilde{\Delta}(n,k,n',k') = \begin{cases} 0, & \text{if } |n-n'| \le 1; \\ \Delta(n,k,n',k'), & \text{otherwise.} \end{cases}$$

Since $\pi_0^*(\operatorname{supp} \chi_{n,k}) \subset \pi_0^*(\operatorname{supp} \widetilde{\chi}_{n,k}) \subset [-2^{n+2}, 2^{n+2}]$, we have that

$$\widetilde{\Delta}(n,k,n',k') \le \Delta(n,k,n',k') \le \max\{n,n'\} - n'/2 + 2 \tag{10}$$

in general. If $|n-n'| \ge 2$ and $\max\{n,n'\} \ge 10$, we have also that

$$\Delta(n,k,n',k') = \widetilde{\Delta}(n,k,n',k') \ge \max\{n,n'\} - n'/2 - 3. \tag{11}$$

Hence it holds, in general, that

$$|n-n'| \le 2\Delta(n,k,n',k') + 10. \tag{12}$$

Remark 7.4. For each $(n,k) \in \mathcal{N}$, the cardinality of $(n',k') \in \mathcal{N}$ such that $\Delta(n,k,n',k') = 0$ (resp. $\Delta(n',k',n,k) = 0$) is bounded by an absolute constant.

Looking into the definition of $\Delta(n,k,n',k')$ more closely, we see that, for each $s > 1$, there exists a constant $C_* > 0$ such that

$$\sum_{(n',k') \in \mathcal{N}} 2^{-s\Delta(n,k,n',k')} < C_* \quad \text{for any } (n,k) \in \mathcal{N} \tag{13}$$

and that

$$\sum_{(n,k)\in\mathcal{N}} 2^{-s\Delta(n,k,n',k')} < C_* \quad \text{for any } (n',k')\in\mathcal{N}. \tag{14}$$

For $(\gamma,\gamma')\in\Gamma$, we write $\Delta(\gamma,\gamma')$ and $\widetilde{\Delta}(\gamma,\gamma')$ respectively for

$$\Delta(n(\gamma),k(\gamma),n(\gamma'),k(\gamma')) \quad \text{and} \quad \widetilde{\Delta}(n(\gamma),k(\gamma),n(\gamma'),k(\gamma')).$$

Lemma 7.5. *For each $\mu>0$, there exists a constant $C_*>0$ such that*

$$|\kappa_{\gamma\gamma'}(x',x)| \le C_*\cdot 2^{-r_*\cdot\Delta(\gamma,\gamma')}\|g\|_*\cdot\int_{Z(\gamma')} b^{\mu}_{\gamma'}(x'-y)\cdot b^{\mu}_{\gamma}(G(y)-x)dy$$

for any $(\gamma,\gamma')\in\Gamma\times\Gamma$ and any $(x,x')\in E\times E$. The constant C_* does not depend on $G:V'\to V$ in $\mathcal{H}$ nor $g\in\mathscr{C}^r(V')$.

Proof. We suppose $\Delta(\gamma,\gamma')>0$, as the conclusion follows from Lemma 7.3 otherwise. Apply the formula (9) to the integral with respect to y in (7), setting $\ell=r_*$, $k=1$ and $\{v_j\}_{j=1}^k=\{v_0\}$. Then we reach the expression

$$\kappa_{\gamma\gamma'}(x',x)=\int\left(\int e^{i\langle\eta,x'-y\rangle-i\langle\xi,G(y)-x\rangle}R(y,\xi,\eta)d\eta d\xi\right)dy \tag{15}$$

where

$$R(y,\xi,\eta)=\frac{i^{r_*}\cdot v_0^{r_*}(\rho_{\gamma'}(y)g(y))\cdot\psi_{\gamma'}(\xi)\cdot\widetilde{\psi}_{\gamma}(\eta)}{(2\pi)^{2(2d+1)}(\pi_0^*(\xi-\eta))^{r_*}}. \tag{16}$$

For any multi-indices $\alpha,\beta\in\mathbb{Z}_+^{2d+1}$, there exists a constant $C_{\alpha\beta}>0$, which is independent of $G:V'\to V$ in $\mathcal{H}$ and $g\in\mathscr{C}^r(V')$, such that

$$\|\partial_\xi^\alpha\partial_\eta^\beta R\|_{L^\infty}\le C_{\alpha,\beta}\cdot\|g\|_*\cdot 2^{-r_*\cdot\Delta(\gamma,\gamma')-|\alpha|n(\gamma)/2-|\alpha|_\dagger|m(\gamma)|-|\beta|n(\gamma')/2-|\beta|_\dagger|m(\gamma')|},$$

where we set $|\alpha|=\alpha_0+\sum_{\sigma=\pm}\sum_{i=1}^d\alpha_i^\sigma$ and $|\alpha|_\dagger=|\alpha|-\alpha_0$ for $\alpha=(\alpha_0,\alpha_1^+,\cdots,\alpha_d^+,\alpha_0^-,\cdots\alpha_d^-)$ and similarly for β. These imply that the integral with respect to ξ and η in the bracket $(\cdot)$ in (15) is bounded by

$$C_*\cdot 2^{-r_*\cdot\Delta(\gamma,\gamma')}\|g\|_*\cdot b^{\mu}_{\gamma'}(x'-y)\cdot b^{\mu}_{\gamma}(G(y)-x)$$

in absolute value. Also the integral vanishes when $y\notin\operatorname{supp}\rho_{\gamma'}\subset Z(\gamma')$. □

8. Preliminaries for the proof of Theorem 7.2

In this section, we give preliminary discussion to the proof of Theorem 7.2. For brevity, we henceforth write $\mathcal{M}$ and $\mathcal{L}$ respectively for $\mathcal{M}(G,g)$ and $\mathcal{L}(G,g)$, though we keep in mind dependence of $\mathcal{M}$ and $\mathcal{L}$ (and many other objects) on G and g.

8.1. **The compact, central and hyperbolic part of $\mathcal{M}$.** In the proof of Theorem 7.2, we divide the operator $\mathcal{M}$ into five parts and consider each parts separately. To this end, we divide the product set $\Gamma \times \Gamma$ into five disjoint subsets $\mathcal{R}(j)$ for $0 \le j \le 4$ and define the corresponding parts $\mathcal{M}_j : \mathbf{B}^{\beta}_{\nu} \to \mathbf{B}^{\beta}_{\nu'}$ of $\mathcal{M}$ formally by

$$\mathcal{M}_j((u_\gamma)_{\gamma\in\Gamma}) = \left(\sum_{\gamma:(\gamma,\gamma')\in\mathcal{R}(j)} \mathcal{L}_{\gamma\gamma'}(u_\gamma) \right)_{\gamma'\in\Gamma}. \tag{17}$$

The definition of the part $\mathcal{M}_0$ is simple. Let $K \ge 0$ be a large constant, which will be determined in the course of the proof, and set

$$\mathcal{R}(0) = \{(\gamma,\gamma') \in \Gamma\times\Gamma \mid \max\{n(\gamma), |m(\gamma)|, n(\gamma'), |m(\gamma')|\} \le K\}.$$

The corresponding part $\mathcal{M}_0$ defined by (17) for $j = 0$ is called the compact part of $\mathcal{M}(G, g)$. This is because we have

Proposition 8.1. *The formal definition of the operator $\mathcal{M}_0$ gives a compact operator $\mathcal{M}_0 : \mathbf{B}^{\beta}_{\nu} \to \mathbf{B}^{\beta}_{\nu'}$ for any $\nu, \nu' \ge 2d + 2$.*

Proof. For $\gamma \in \Gamma$, let $L^2(E; d^{\nu}_{\gamma})$ be the Hilbert space of functions $u \in L^2(E)$ such that $\|d^{\nu}_{\gamma} \cdot u_\gamma\|_{L^2} < \infty$, equipped with the obvious norm. Then

$$\mathcal{L}_{\gamma\gamma'} : L^2(E; d^{\nu}_{\gamma}) \to L^2(E; d^{\nu'}_{\gamma'}) \tag{18}$$

is a compact operator, because its kernel (7) is smooth and decay rapidly as we saw in Lemma 7.3. Since $\mathcal{R}(0)$ contains only finitely many elements, the statement follows immediately. □

The part $\mathcal{M}_0$ will turn out to be the compact operator $\mathcal{K}(G, g)$ in the latter statement of Theorem 7.2.

The definition of the part $\mathcal{M}_1$ is not involved. Let $0 < \delta < 1/10$ be a constant that we will fix soon below. For given $\lambda > 0$, we set

$$\begin{aligned}\mathcal{R}(1) &= \mathcal{R}(1;\lambda)\\ &= \{(\gamma,\gamma') \in \Gamma\times\Gamma \setminus \mathcal{R}(0) \mid \max\{|m(\gamma)|, |m(\gamma')|\} \le \delta\lambda,\ |n(\gamma) - n(\gamma')| \le 1\}.\end{aligned}$$

The corresponding part $\mathcal{M}_1$ is called the central part of $\mathcal{M}$. The remaining part is called hyperbolic part and will be divided into three parts.

8.2. **Setting of constants.** In the proof, we set up the constants as follows. We henceforth suppose that $0 < \beta < (r - 1)/2$ and $\epsilon > 0$ in the statement of Theorem 7.2 are given. Then we first choose $0 < \delta < 1/10$ so small that

$$(2\beta + 5d + 2)\delta < \epsilon.$$

Next we choose ν_*, λ_* and Λ_* in the conclusion of Theorem 7.2 so large that

$$\nu_* \ge 6(\beta/\delta + d + 1)$$

and that

$$\lambda_* > 40, \quad 2^{\delta\lambda_* - 10} \ge 10^2\sqrt{2d+1}, \quad \Lambda_* \ge d\lambda_*.$$

Note that the conditions above are technical and the readers should not care about them too much at this stage. We present them only to emphasize that the choices are explicit.

Once we set up the constants δ, ν_*, λ_* and Λ_* as above, we take $\lambda \geq \lambda_*$ and $\Lambda \geq \Lambda_*$ such that $\Lambda \geq d\lambda$ and then take an arbitrary diffeomorphism $G : V' \to V$ in $\mathcal{H}(\lambda, \Lambda)$ and an arbitrary function g in $\mathscr{C}^r(V')$. This is the setting in which most of the argument in the following sections is developed.

The readers should be aware that the choice of the constant $K > 0$ in the definition of $\mathcal{R}(0)$ is not mentioned above. We will choose the constant K in the course of the proof and the choice will depend on the diffeomorphism G and the function g besides λ and Λ. This does not cause any problem because Proposition 8.1 holds regardless of the choice of K. In the proof, we understand that the constant K is taken so large that the argument holds true and will not mention the choice of K too often.

In the proof, it is important to distinguish the class of constants that are independent of the diffeomorphism $G : V' \to V$ in $\mathcal{H}(\lambda, \Lambda)$ and the function $g : V' \to \mathbb{R}$ in $\mathscr{C}^r(V')$ and also of the choice of λ and Λ. To this end, we use a generic symbol C_* for such class of constants. On the contrary, we use the generic symbol $C(G, g)$ (resp. $C(G)$) for constants that may depend on G and g (resp. on G) and also on λ and Λ. Notice that the real value of constants denoted by C_*, $C(G, g)$ and $C(G)$ may change places to places in the argument.

8.3. **Norms on $\mathbf{B}^{\beta}_{\nu}$.** In the proof, we consider the following family of norms on $\mathbf{B}^{\beta}_{\nu}$ for $\lambda > 0$, rather than the original norm $\|\cdot\|_{\beta,\nu}$ in the definition:

$$\|\mathbf{u}\|^{(\lambda)}_{\beta,\nu} = \left(\sum_{\gamma} w^{(\lambda)}(m(\gamma))^2 \cdot \|d^{\nu}_{\gamma} \cdot u_{\gamma}\|^2_{L^2} \right)^{1/2} \qquad \text{for } \mathbf{u} = (u_\gamma)_{\gamma \in \Gamma},$$

where

$$w^{(\lambda)}(m) = \begin{cases} 2^{\beta(m+2\lambda)}, & \text{if } m > \delta\lambda; \\ 1, & \text{if } |m| \leq \delta\lambda; \\ 2^{\beta(m-2\lambda)}, & \text{if } m < -\delta\lambda. \end{cases}$$

This family of norms are all equivalent to the original norm $\|\cdot\|_{\beta,\nu}$ because

$$2^{\beta(m-2\lambda)} \leq w^{(\lambda)}(m) \leq 2^{\beta(m+2\lambda)}. \tag{19}$$

The family of norms $\|\cdot\|^{(\lambda)}_{\beta,\nu_*}$ will turn out to be the norms $\|\cdot\|^{(\lambda)}$ in the latter statement of Theorem 7.2.

9. The hyperbolic parts of the operator $\mathcal{M}$ (I)

In this section and the following two sections, we consider the hyperbolic part of the operator $\mathcal{M}$. We divide it into three parts, namely, $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_4$, and estimate the operator norms of each part separately. The rough idea in this division is as follows. From the definition of the operator $\mathcal{L}_{\gamma\gamma'}$, we naturally expect that the operator norm of $\mathcal{L}_{\gamma\gamma'}$ should be small if either

(A) $G(z(\gamma'))$ is apart from $z(\gamma)$, or

(B) $DG_y^*(\operatorname{supp}\widetilde{\psi}_\gamma)$ for $y \in \operatorname{supp}\rho_{\gamma'}$ are apart from $\operatorname{supp}\psi_{\gamma'}$.

Roughly, $\mathcal{M}_3$ and $\mathcal{M}_4$ consist of the components $\mathcal{L}_{\gamma\gamma'}$ for pairs (γ, γ') in the cases (A) and (B) respectively. And we will in fact prove that the operator norms of $\mathcal{M}_3$ and $\mathcal{M}_4$ are small in Section 10 and 11. The remaining components $\mathcal{L}_{\gamma\gamma'}$ are assigned to the part $\mathcal{M}_2$, which gives raise to the factor $2^{-\beta\lambda}$ in the claim of Theorem 7.2.

9.1. **The operator $\mathcal{M}_2$.** We first define the operator $\mathcal{M}_2$ as follows.

Definition 9.1. Let $\mathcal{R}(2)$ be the set of pairs $(\gamma, \gamma') \in \Gamma \times \Gamma \setminus (\mathcal{R}(0) \cup \mathcal{R}(1))$ such that $n = n(\gamma)$, $k = k(\gamma)$, $m = m(\gamma)$, $n' = n(\gamma')$, $k' = k(\gamma')$ and $m' = m(\gamma')$ satisfy either of the following conditions:

(a) $m' < m - \lambda + 10\widetilde{\Delta}(n, k, n', k') + 20$, or

(b) $|n - n'| \le 1$ and either $m' < -\delta\lambda \le m$ or $m' \le \delta\lambda < m$.

Let $\mathcal{M}_2$ be the part defined formally by (17) for $j = 2$.

Proposition 9.2. *The formal definition of the operator $\mathcal{M}_2$ in fact gives a bounded operator $\mathcal{M}_2 : \mathbf{B}^\beta_\nu \to \mathbf{B}^\beta_{\nu'}$ for any $\nu, \nu' \ge 2d + 2$. Further, for any $\nu, \nu' \ge 2d + 2$, there is a constant $C_* > 0$ such that we have*

$$\|\mathcal{M}_2(\mathbf{u})\|^{(\lambda)}_{\beta,\nu'} \le C_* \cdot \|g\|_* \cdot 2^{-\beta\lambda} \cdot \|\mathbf{u}\|^{(\lambda)}_{\beta,\nu} \quad \text{for } \mathbf{u} \in \mathbf{B}^\beta_\nu,$$

for $G : V' \to V$ in $\mathcal{H}(\lambda, \Lambda)$ and $g \in \mathscr{C}^r(V')$ provided $\lambda \ge \lambda_*$ and $\Lambda \ge \Lambda_*$.

Proof. For a combination $(n, k, m, n', k', m') \in (\mathcal{N} \oplus \mathbb{Z})^2$, we set

$$K_{n,k,m,n',k',m'} = 2^{-r_* \cdot \Delta(n,k,n',k')} \cdot \|g\|_* \cdot \frac{w^{(\lambda)}(m')}{w^{(\lambda)}(m)}. \tag{20}$$

We need the following sublemma of combinatorial nature, whose proof is postponed for a while.

Sublemma 9.3. *There exists a constant $C_* > 0$ such that*

$$\sup_{(n',k',m') \in \mathcal{N} \oplus \mathbb{Z}} \left(\sum_{n,k,m:n',k',m'} K_{n,k,m,n',k',m'} \right) < C_* \|g\|_* \cdot 2^{-\beta\lambda} \tag{21}$$

and

$$\sup_{(n,k,m) \in \mathcal{N} \oplus \mathbb{Z}} \left(\sum_{n',k',m':n,k,m} K_{n,k,m,n',k',m'} \right) < C_* \|g\|_* \cdot 2^{-\beta\lambda} \tag{22}$$

where $\sum_{n',k',m':n,k,m}$ (resp. $\sum_{n,k,m:n',k',m'}$) *denotes the sum over (n', k', m')* (resp. (n, k, m)) *in $\mathcal{N} \oplus \mathbb{Z}$ such that (n, k, m, n', k', m') satisfies*

$$\max\{n, |m|, n', |m'|\} > K, \tag{23}$$

and either of the conditions (a) or (b) in the definition of $\mathcal{R}(2)$.

For $(n,k,m)\in\mathcal{N}\oplus\mathbb{Z}$, we set

$$v_{n,k,m}(x)=\left(\sum_{\gamma:n,k,m} d_\gamma^{2\nu}(x)\cdot|u_\gamma(x)|^2\right)^{1/2} \tag{24}$$

where $\sum_{\gamma:n,k,m}$ denotes the sum over $\gamma\in\Gamma$ such that $n(\gamma)=n$, $k(\gamma)=k$ and $m(\gamma)=m$. Then we have, by Schwarz inequality, that

$$\sum_{\gamma:n,k,m}|u_\gamma(x)|\le\left(\sum_{\gamma:n,k,m} d_\gamma^{-2\nu}(x)\right)^{1/2}\cdot v_{n,k,m}(x)\le C_*\cdot v_{n,k,m}(x). \tag{25}$$

From Lemma 7.5 with $\mu=\nu'+2d+2$, we have the following estimate on the kernel $\kappa_{\gamma\gamma'}$ of the operator $\mathcal{L}_{\gamma\gamma'}$:

$$|d_{\gamma'}^{\nu'}(x')\cdot\kappa_{\gamma\gamma'}(x',x)|\le C_*\cdot 2^{-r_*\cdot\Delta(\gamma,\gamma')}\|g\|_*\cdot\int_{Z(\gamma')} b_{\gamma'}^{2d+2}(x'-y)\cdot b_\gamma^{\mu}(G(y)-x)dy.$$

Hence, by Young inequality, we obtain

$$\left\|\sum_{\gamma:n,k,m} d_{\gamma'}^{\nu'}\mathcal{L}_{\gamma\gamma'}u_\gamma\right\|_{L^2}\le C_*\left|K_{n,k,m,n',k',m'}\frac{w^{(\lambda)}(m)}{w^{(\lambda)}(m')}\right|\left\|b_{n,m}^{\mu}*v_{n,k,m}|_{G(Z(\gamma'))}\right\|_{L^2}$$

for $\gamma'\in\Gamma$ such that $n(\gamma')=n'$, $k(\gamma')=k'$ and $m(\gamma')=m'$. Since the intersection multiplicity of $Z(\gamma')$ for $\gamma'\in\Gamma$ such that $n(\gamma')=n'$, $k(\gamma')=k'$ and $m(\gamma')=m'$ is bounded by some constant depending only on d, it follows

$$\sum_{\gamma':n',k',m'}\left\|\sum_{\gamma:n,k,m} d_{\gamma'}^{\nu'}\mathcal{L}_{\gamma\gamma'}u_\gamma\right\|_{L^2}^2\le C_*\left|K_{n,k,m,n',k',m'}\frac{w^{(\lambda)}(m)}{w^{(\lambda)}(m')}\right|^2\|v_{n,k,m}\|_{L^2}^2.$$

By definition, the square of $\|\mathcal{M}_2(\mathbf{u})\|_{\beta,\nu'}^{(\lambda)}$ for $\mathbf{u}=(u_\gamma)_{\gamma\in\Gamma}\in\mathbf{B}_\nu^\beta$ equals

$$\sum_{n',k',m'}\sum_{\gamma':n',k',m'} w^{(\lambda)}(m')^2\left\|\sum_{n,k,m:n',k',m'}\sum_{\gamma:n,k,m} d_{\gamma'}^{\nu'}\mathcal{L}_{\gamma\gamma'}u_\gamma\right\|_{L^2}^2.$$

Using Schwarz inequality, the inequality above and also (21) and (22), we see that this is bounded by

$$\begin{aligned}&\sum_{n',k',m'}\sum_{n,k,m:n',k',m'}\frac{C_*\|g\|_*\cdot 2^{-\beta\lambda}\cdot w^{(\lambda)}(m')^2}{K_{n,k,m,n',k',m'}}\sum_{\gamma':n',k',m'}\left\|\sum_{\gamma:n,k,m} d_{\gamma'}^{\nu'}\mathcal{L}_{\gamma\gamma'}u_\gamma\right\|_{L^2}^2\\
&\le\sum_{n,k,m}\sum_{n',k',m':n,k,m} C_*\|g\|_*\cdot 2^{-\beta\lambda}\cdot K_{n,k,m,n',k',m'}\cdot w^{(\lambda)}(m)^2\cdot\|v_{n,k,m}\|_{L^2}^2\\
&\le C_*\|g\|_*^2\cdot 2^{-2\beta\lambda}\cdot\sum_{n,k,m} w^{(\lambda)}(m)^2\cdot\|v_{n,k,m}\|_{L^2}^2\le C_*\|g\|_*^2\cdot 2^{-2\beta\lambda}\cdot(\|\mathbf{u}\|_{\beta,\nu}^{(\lambda)})^2.\end{aligned}$$

We now complete the proof by proving Sublemma 9.3.

Proof of Sublemma 9.3. In the argument below, we consider combinations $(n,k,m,n',k',m') \in (\mathcal{N}\times\mathbb{Z})^2$ satisfying (23) and either of the conditions (a) or (b) in the definition of $\mathcal{R}(2)$. And we further restrict our attention to those combinations in the cases (I) $|n-n'|\le 1$ and (II) $|n-n'|\ge 2$ in turn and prove the claims (21) and (22) with the sums replaced by the partial sums restricted to such cases. This is of course enough for the proof of the sublemma.

Case (I): If the condition (a) in the definition of $\mathcal{R}(2)$ holds in addition, we have $m' < m-\lambda+20 < m-2\delta\lambda$ from the choice of δ and λ_*, and hence

$$K_{n,k,m,n',k',m'} \le 2^{\beta(m'-m)-r_*\Delta(n,k,n',k')}\|g\|_*. \tag{26}$$

Otherwise the condition (b) holds and hence we have $m'<m$ and

$$K_{n,k,m,n',k',m'} \le 2^{-2\beta\lambda+\beta(m'-m-2\delta\lambda)-r_*\Delta(n,k,n',k')}\|g\|_*. \tag{27}$$

Therefore, considering each of these two subcases separately and using (13) and (14), we obtain the required inequalities for the partial sums.

Case (II): Notice that the condition (a) in the definition of $\mathcal{R}(2)$ holds for the combinations under consideration in this case. If $\max\{n,n'\}\le K/100$ and if m and m' are on the same side of the interval $[-\delta\lambda,\delta\lambda]$, we have (26), which can be written as

$$K_{n,k,m,n',k',m'} \le 2^{\beta(m'-m-10\Delta(n,k,n',k'))-(r_*-10\beta)\Delta(n,k,n',k')}\|g\|_*.$$

If $\max\{n,n'\}\le K/100$ and if m and m' are not on the same side of $[-\delta\lambda,\delta\lambda]$, we have $m'-m\le -K/2$ because $\max\{|m|,|m'|\}\ge K$ from (23) and because $\widetilde{\Delta}(n,k,n',k')\le K/50$ from (10). If we have $\max\{n,n'\}>K/100$, it holds $\widetilde{\Delta}(n,k,n',k')\ge K/200-3$ from (11). Considering each of the three subcases above separately and using (13), (14) and also the general estimate

$$K_{n,k,m,n',k',m'} \le 2^{\beta(m'-m)-r_*\Delta(n,k,n',k')+4\beta\lambda}\|g\|_*$$

in the latter two subcases, we obtain the required inequalities for the partial sums, provided that we take sufficiently large constant K. □

9.2. **A dichotomy in the remaining case.** In this subsection, we prove a lemma which tells roughly that each pair (γ,γ') that belongs to neither of $\mathcal{R}(j)$ for $j=0,1,2$ falls into either of the situation (A) or (B) mentioned in the beginning of this section. First of all, we note that a pair $(\gamma,\gamma')\in\Gamma\times\Gamma$ belongs to neither of $\mathcal{R}(0)$, $\mathcal{R}(1)$ or $\mathcal{R}(2)$ if and only if $n=n(\gamma)$, $k=k(\gamma)$, $m=m(\gamma)$, $n'=n(\gamma')$, $k'=k(\gamma')$ and $m'=m(\gamma')$ satisfy the conditions

- (R1) $\max\{n,n',|m|,|m'|\}>K$,
- (R2) $\max\{|m|,|m'|\}>\delta\lambda$ if $|n-n'|\le 1$,
- (R3) $m'\ge m-\lambda+10\widetilde{\Delta}(n,k,n',k')+20$, and
- (R4) neither $m'<-\delta\lambda\le m$ nor $m'\le\delta\lambda<m$ if $|n-n'|\le 1$.

For convenience, we list the following immediate consequences of (R1)-(R4):

- (R5) $m'\ge m-\lambda+20$,
- (R6) either $m<0$ or $m'>0$,

(R7) if $|n-n'|\le 1$, we have $\max\{-m,m'\}\ge \delta\lambda$,
(R8) if $|n-n'|\ge 2$, we have

$$\max\{-m,m'\}\ge 2\max\{n,n'\} \quad\text{and}\quad \max\{-m,m'\}\ge K/100.$$

Clearly (R5) follows from (R3), and (R6) follows from (R7) and (R8). (R7) follows from (R2) and (R4). If $\max\{n,n'\}\ge K/100$, (R8) follows from (R3) and (11). Otherwise we have $\max\{|m|,|m'|\}\ge K$ from (R1) and hence $\max\{-m,m'\}\ge K/2$ from (R5), which implies (R8).

Next we give a few definitions in order to state the next lemma. For a pair $(\gamma,\gamma')\in\Gamma\times\Gamma$ that belongs to neither of $\mathcal{R}(j)$ for $j=0,1,2$, we set

$$D(\gamma,\gamma')=D(n,m,n',m') \quad\text{and}\quad \widetilde{D}(\gamma,\gamma')=\widetilde{D}(n,m,n',m')$$

where $n=n(\gamma)$, $m=m(\gamma)$, $n'=n(\gamma')$ and $m'=m(\gamma')$ and[4]

$$D(n,m,n',m')=\begin{cases} m'+n'/2, & \text{if } m\ge 0,\ m'>0;\\ -m+n/2+\lambda, & \text{if } m<0,\ m'<0;\\ \max\{-m+n/2+\lambda, m'+n'/2\}, & \text{if } m<0,\ m'\ge 0,\end{cases}$$

and

$$\widetilde{D}(n,m,n',m')=\begin{cases} m'+n'/2-n+\lambda, & \text{if } m\ge 0,\ m'>0;\\ -m-n/2, & \text{if } m<0,\ m'<0;\\ \max\{-m-n/2, m'+n'/2-n+\lambda\}, & \text{if } m<0,\ m'\ge 0.\end{cases}$$

Let $\Pi_z : E^*\to E^*_+\oplus E^*_-$ be the projection along the line $\langle\alpha_0(z)\rangle$ spanned by $\alpha_0(z)$. Then we have, from the definition of α_0, that

$$\|\Pi_z(\xi)-\Pi_{z'}(\xi)\| = |\pi_0^*(\xi)|\cdot\|z-z'\| \qquad \text{for } \xi\in E^* \text{ and } z,z'\in E. \tag{28}$$

Lemma 9.4. *If $d(G(Z(\gamma')),z(\gamma))\le 2^{\widetilde{D}(\gamma,\gamma')-10}$ for a pair $(\gamma,\gamma')\in\Gamma\times\Gamma$ that belongs to neither of $\mathcal{R}(j)$ for $j=0,1,2$, we have*

$$d(\Pi_{z(\gamma')}(\operatorname{supp}\psi_{\gamma'}), \Pi_{z(\gamma')}(DG^*_y(\operatorname{supp}\widetilde{\psi}_\gamma)))\ge 2^{D(\gamma,\gamma')-10} \tag{29}$$

for all $y\in Z(\gamma')$. Further, if

$$\max\{|m(\gamma)|,|m(\gamma')|\}\le \max\{n(\gamma),n(\gamma')\}/4 \tag{30}$$

in addition, we have (29) for all $y\in E$ such that $\|y-z(\gamma')\|<2^{-n(\gamma)/3}$.

Proof of Lemma 9.4. Take $(\gamma,\gamma')\in(\Gamma\times\Gamma)\setminus\cup_{j=0}^2\mathcal{R}(j)$ and set $n=n(\gamma)$, $k=k(\gamma)$, $m=m(\gamma)$, $n'=n(\gamma')$, $k'=k(\gamma')$ and $m'=m(\gamma')$. We first prove

Sublemma 9.5. *If $w\in Z(\gamma')$ satisfies $d(G(w),z(\gamma))\le 2^{\widetilde{D}(n,m,n',m')-8}$, we have that $d(\Pi_{z(\gamma')}(\operatorname{supp}\psi_{\gamma'}), \Pi_{z(\gamma')}(DG^*_w(\operatorname{supp}\widetilde{\psi}_\gamma)))\ge 2^{D(n,m,n',m')-8}$.*

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[4]Because of (R6), we do not consider the case ($m\ge 0$ and $m'\le 0$).

Proof. We prove the claim only in the case $m \geq 0$ and $m' \geq 0$, because the proofs in the other cases are similar. By (28), the Hausdorff distance between $\Pi_{G(w)}(\operatorname{supp} \widetilde{\psi}_\gamma)$ and $\Pi_{z(\gamma)}(\operatorname{supp} \widetilde{\psi}_\gamma)$ is bounded by

$$2^{n+2} \cdot d(G(w), z(\gamma)) \leq 2^{n+2+\widetilde{D}(n,m,n',m')-8} = 2^{m'+n'/2+\lambda-6}.$$

Hence the subset $\Pi_{G(w)}(\operatorname{supp} \widetilde{\psi}_\gamma)$ is contained in the disk $\mathbb{D}^*_{+,-}(R)$ in the subspace $E^*_+ \oplus E^*_-$ with center at the origin and radius

$$R = 2^{m'+n'/2+\lambda-5} \geq 2^{m+n/2+2} + 2^{m'+n'/2+\lambda-6}$$

where the inequality is a consequence of the condition (R3) and (12). From the definition of $\mathcal{H}(\lambda, \Lambda)$, it implies that the subset $\Pi_w(DG^*_w(\operatorname{supp} \widetilde{\psi}_\gamma))$ is contained in $\mathbb{D}^*_{+,-}(2^{-\lambda}R) \cup \mathbf{C}^*_-(1/10)$. Again by (28), the Hausdorff distance between $\Pi_{z(\gamma')}(DG^*_w(\operatorname{supp} \widetilde{\psi}_\gamma))$ and $\Pi_w(DG^*_w(\operatorname{supp} \widetilde{\psi}_\gamma))$ is bounded by

$$2^{n+2} \cdot d(w, z(\gamma')) \leq \sqrt{2d+1} \cdot 2^{n-n'/2+3}.$$

Hence $\Pi_{z(\gamma')}(DG^*_w(\operatorname{supp} \widetilde{\psi}_\gamma))$ is contained in $\mathbb{D}^*_{+,-}(R') \cup \mathbf{C}^*_-(2/10)$ where

$$R' = 2^{-\lambda}R + 10^2\sqrt{2d+1} \cdot 2^{n-n'/2+3}.$$

On the other hand, the subset $\Pi_{z(\gamma')}(\operatorname{supp} \psi_{\gamma'})$ is contained in the cone $\mathbf{C}^*_+(6/10)$ and bounded away from $\mathbb{D}^*_{+,-}(2^{m'+n'/2-1}) = \mathbb{D}^*_{+,-}(2^{-\lambda+4}R)$ by definition. Thus the claim follows if we prove

$$10^2\sqrt{2d+1} \cdot 2^{n-n'/2+3} \leq 2^{-\lambda}R = 2^{m'+n'/2-5}.$$

If $|n-n'| \leq 1$, this follows from (R7) and the choice of λ_*. Othewise this follows from (R8), provided that K is sufficiently large. □

Now we prove Lemma 9.4 by using the sublemma above. Let us first consider the case where (30) holds. Note that we have $\max\{n, n'\} \geq K$ from (R1) and $|n'-n| \leq 1$ from (R8). Corollary 4.2 tells that

$$\|DG^*_y(\xi) - DG^*_{z(\gamma')}(\xi)\| < C(G,g)2^{(5/12)\max\{n,n'\}} < 2^{D(n,m,n',m')-10}$$

for $\xi \in \operatorname{supp} \widetilde{\psi}_\gamma$ and $y \in E$ such that $d(y, z(\gamma')) < 2^{-n/3}$. Clearly the claim of the lemma follows from this and the sublemma.

Next we consider the case where (30) does *not* hold. By virtue of the sublemma, it is enough to show

$$\operatorname{diam} G(Z(\gamma')) \leq 2^{\widetilde{D}(n,m,n',m')-10}. \tag{31}$$

If $|n-n'| \leq 1$, we have $\max\{-m, m'\} \geq K/5$ from (R1) and (R5), and hence (31) follows provided that we take large K according to G. Otherwise (31) follows from (R8). □

10. The hyperbolic parts of the operator $\mathcal{M}$ (II)

Let $\mathcal{R}(3)$ be the set of pairs $(\gamma,\gamma')\in\Gamma\times\Gamma\setminus\cup_{i=0}^{2}\mathcal{R}(i)$ such that

$$d(G(Z(\gamma')),z(\gamma))>2^{\widetilde{D}(\gamma,\gamma')-10}. \tag{32}$$

We consider the part $\mathcal{M}_3$ defined formally by (17) for $j=3$ and prove

Proposition 10.1. *The formal definition of the operator $\mathcal{M}_3$ in fact gives a bounded operator $\mathcal{M}_3:\mathbf{B}^{\beta}_{\nu}\to\mathbf{B}^{\beta}_{\nu'}$ for any $\nu,\nu'\geq 2\beta+2d+2$. Further there is a constant $C_*>0$ such that we have*

$$\|\mathcal{M}_3(\mathbf{u})\|^{(\lambda)}_{\beta,\nu_*}\leq C_*\|g\|_{L^\infty}\cdot 2^{-\beta\lambda}\cdot\|\mathbf{u}\|^{(\lambda)}_{\beta,\nu_*}\qquad\text{for }\mathbf{u}\in\mathbf{B}^{\beta}_{\nu_*}$$

for $G:V'\to V$ in $\mathcal{H}(\lambda,\Lambda)$ and $g\in\mathscr{C}^r(V')$ provided $\lambda\geq\lambda_*$ and $\Lambda\geq\Lambda_*$.

Proof. The structure of the proof is similar to that of Proposition 9.2, though we consider combinations (n,k,m,n',k',m') in $(\mathcal{N}\oplus\mathbb{Z})^2$ that satisfy the conditions (R1)-(R4). For this time, we set

$$K_{n,k,m,n',k',m'}=2^{-(\nu-2d-2)\cdot(\widetilde{D}(n,m,n',m')+n/2)-r_*\Delta(n,k,n',k')}\cdot\|g\|_*\cdot\frac{w^{(\lambda)}(m')}{w^{(\lambda)}(m)}.$$

And we use the following sublemma of combinatorial nature in the place of Sublemma 9.3, whose proof is postponed for a while.

Sublemma 10.2. *There exists a constant $C_*>0$ such that*

$$\sup_{(n',k',m')\in\mathcal{N}\oplus\mathbb{Z}}\left(\sum_{n,k,m|n',k',m'}K_{n,k,m,n',k',m'}\right)<C_*\|g\|_*\lambda\cdot 2^{-(\nu-2\beta-2d-2)\delta\lambda+4\beta\lambda}$$

and

$$\sup_{(n,k,m)\in\mathcal{N}\oplus\mathbb{Z}}\left(\sum_{n',k',m'|n,k,m}K_{n,k,m,n',k',m'}\right)<C_*\|g\|_*\lambda\cdot 2^{-(\nu-2\beta-2d-2)\delta\lambda+4\beta\lambda}$$

where $\sum_{n',k',m'|n,k,m}$ (resp. $\sum_{n,k,m|n',k',m'}$) *denotes the sum over (n',k',m')* (resp. (n,k,m)) *in $\mathcal{N}\oplus\mathbb{Z}$ such that (n,k,m,n',k',m') satisfies (R1)-(R4).*

By the definition of $\mathcal{R}(3)$, we have, for $(\gamma,\gamma')\in\mathcal{R}(3)$ and $y\in Z(\gamma')$,

$$\sup_{x\in E}\left(\langle 2^{n(\gamma)/2}(G(y)-x)\rangle^{-1}\cdot d_\gamma(x)^{-1}\right)\leq 2^{-\widetilde{D}(\gamma,\gamma')-n(\gamma)/2+11}.$$

Hence, from Lemma 7.5 for $\mu=\max\{\nu,\nu'\}+4d+4$, we have that

$$\begin{aligned}|d^{\nu'}_{\gamma'}(x')\kappa_{\gamma\gamma'}(x',x)d_\gamma^{-\nu+2d+2}(x)|&\\ \leq C_*\|g\|_*\cdot 2^{-(\nu-2d-2)(\widetilde{D}(n,m,n',m')+n/2)-r_*\Delta(n,k,n',k')}&\\ \cdot\int_{Z(\gamma')}b^{2d+2}_{\gamma'}(x'-y)\cdot b^{\mu-\nu+2d+2}_{\gamma}(G(y)-x)dy&\end{aligned}$$

for $(\gamma, \gamma') \in \mathcal{R}(3)$, where $n = n(\gamma)$, $k = k(\gamma)$, $m = m(\gamma)$, $n' = n(\gamma')$, $k' = k(\gamma')$ and $m' = m(\gamma')$. This estimate and Young inequality yield

$$\sum_{\gamma':n',k',m'} \left\| {\sum_{\gamma:n,k,m;\gamma'}}^{\dagger} d_{\gamma'}^{\nu'} \mathcal{L}_{\gamma\gamma'} u_\gamma \right\|_{L^2}^2 \le C_* \cdot \left| K_{n,k,m,n',k',m'} \cdot \frac{w^{(\lambda)}(m)}{w^{(\lambda)}(m')} \right|^2 \cdot \left\| \sum_{\gamma:n,k,m} d_\gamma^{\nu-2d-2} u_\gamma \right\|_{L^2}^2$$

for $\mathbf{u} = (u_\gamma)_{\gamma\in\Gamma} \in \mathbf{B}_\nu^\beta$, where $\sum_{\gamma:n,k,m;\gamma'}^{\dagger}$ denotes the sum over $\gamma \in \Gamma$ such that $n(\gamma) = n$, $k(\gamma) = k$ and $m(\gamma) = m$ and that $(\gamma, \gamma') \in \mathcal{R}(3)$, while $\sum_{\gamma:n,k,m}$ denotes the sum over $\gamma \in \Gamma$ such that $n(\gamma) = n$, $k(\gamma) = k$ and $m(\gamma) = m$. Applying Schwarz inequality as in (25), we get

$$\sum_{\gamma':n',k',m'} \left\| {\sum_{\gamma:n,k,m:\gamma'}}^{\dagger} d_{\gamma'}^{\nu'} \mathcal{L}_{\gamma\gamma'} u_\gamma \right\|_{L^2}^2 \le C_* \left| K_{n,k,m,n',k',m'} \frac{w^{(\lambda)}(m)}{w^{(\lambda)}(m')} \right|^2 \|v_{n,k,m}\|_{L^2}^2$$

where $v_{n,k,m}$ is defined by (24). Once we have this estimate, we can proceed just as in the last part of the proof of Proposition 9.2, using Sublemma 10.2 in the place of Sublemma 9.3, and conclude that

$$\|\mathcal{M}_3(\mathbf{u})\|_{\beta,\nu'}^{(\lambda)} \le C_* \|g\|_* \cdot \lambda \cdot 2^{-(\nu-2\beta-2d-2)\delta\lambda+4\beta\lambda} \cdot \|\mathbf{u}\|_{\beta,\nu}^{(\lambda)} \quad \text{for } \mathbf{u} \in \mathbf{B}_\nu^\beta.$$

This implies not only that $\mathcal{M}_3 : \mathbf{B}_\nu^\beta \to \mathbf{B}_{\nu'}^\beta$ is bounded but also the latter claim of the proposition because $-(\nu_* - 2\beta - 2d - 2)\delta\lambda + 4\beta\lambda < -\beta\lambda$ from the choice of ν_*. We finish the proof by proving Sublemma 10.2.

Proof of Sublemma 10.2. In the argument below, we consider combinations (n, k, m, n', k', m') in $(\mathcal{N} \times \mathbb{Z})^2$ that satisfy the conditions (R1)-(R4). Recalling the definition of $\widetilde{D}(n, m, n', m')$, we see

$$\widetilde{D}(n, m, n', m') + n/2 + |n - n'|/2 \ge \max\{-m, m'\}.$$

Also we have, by (12), that

$$2\widetilde{D}(n, m, n', m') + n + 2\Delta(n, k, n', k') + 10 \ge 2\max\{-m, m'\} \ge m' - m.$$

This and (19) imply

$$K_{n,k,m,n',k',m'} \le C_* 2^{-(\nu-2\beta-2d-2)(\widetilde{D}(n,m,n',m')+n/2)-(r_*-2\beta)\Delta(n,k,n',k')+4\beta\lambda}.$$

Below we proceed as in the proof of Sublemma 9.3: We restrict our attention to the cases (I) $|n' - n| \le 1$ and (II) $|n' - n| \ge 2$ in turn, and prove the claims with the sums replaced by the partial sums restricted to such cases.

Case (I): In this case, the estimates above imply that

$$K_{n,k,m,n',k',m'} \le C_* 2^{-(\nu-2\beta-2d-2)\max\{-m,m'\}-(r_*-2\beta)\Delta(n,k,n',k')+4\beta\lambda}. \tag{33}$$

Therefore, taking (R5) and (R7) and also (13) and (14) into consideration, we obtain the required estimates for the partial sums.

Case (II): In this case, we have

$$\widetilde{D}(n,m,n',m') + n/2 \geq \max\{-m,m'\} - \max\{n,n'\} \geq \max\{-m,m'\}/2$$

from (R8). Hence it follows from the estimates above that

$$K_{n,k,m,n',k',m'} \leq C_* 2^{-(\nu-2\beta-2d-2)\max\{-m,m'\}/2-(r_*-2\beta)\Delta(n,k,n',k')+4\beta\lambda}. \tag{34}$$

Using this estimate and taking (R5) and (R8) and also (13) and (14) into consideration, we obtain the required inequalities for the partial sums. □

11. The hyperbolic parts of the operator $\mathcal{M}$ (III)

In this section we consider the remainder of the hyperbolic part. We set $\mathcal{R}(4) = \Gamma \times \Gamma \setminus (\cup_{i=0}^{3} \mathcal{R}(i))$ and let $\mathcal{M}_4$ be the part defined formally by (17) for $j = 4$. We prove

Proposition 11.1. *The formal definition of the operator $\mathcal{M}_4$ in fact gives a bounded operator $\mathcal{M}_4 : \mathbf{B}^{\beta}_{\nu} \to \mathbf{B}^{\beta}_{\nu'}$ for $\nu, \nu' \geq 2d+2$. Further, there exists a constant $C_* > 0$ such that we have*

$$\|\mathcal{M}_4(\mathbf{u})\|^{(\lambda)}_{\beta,\nu_*} \leq C_* \cdot \|g\|_* \cdot 2^{-\beta\lambda} \|\mathbf{u}\|^{(\lambda)}_{\beta,\nu_*} \quad \text{for } \mathbf{u} \in \mathbf{B}^{\beta}_{\nu}$$

for $G : V' \to V$ in $\mathcal{H}(\lambda, \Lambda)$ and $g \in \mathscr{C}^r(V')$ provided $\lambda \geq \lambda_*$ and $\Lambda \geq \Lambda_*$.

In the proof, we need the following estimate on the kernel $\kappa_{\gamma\gamma'}$ of $\mathcal{L}_{\gamma\gamma'}$. This is a key lemma in our argument on the hyperbolic part.

Lemma 11.2. *For $\mu \geq 2d+2$ and $\mu' > 0$, there exist a constant $C_* > 0$, and another constant $C(G,g)$ that may depend on G and g, such that*

$$|\kappa_{\gamma\gamma'}(x',x)| \leq C(n(\gamma), m(\gamma), n(\gamma'), m(\gamma')) \cdot 2^{-r_* \cdot \Delta(\gamma,\gamma')} \int b^{\mu}_{\gamma'}(x',y) d^{-2d-2}_{\gamma'}(y) b^{\mu}_{\gamma}(G(y),x) dy$$

for $(\gamma, \gamma') \in \mathcal{R}(4)$ and $x, x' \in E$, where we set

$$C(n,m,n',m') = C_* \|g\|_* 2^{-\mu'(D(n,m,n',m')-n'/2)} + C(G,g) 2^{-(r-1)(D(n,m,n',m')-n'/3)}$$

in the case (30) holds and, otherwise,

$$C(n,m,n',m') = C(G,g) 2^{-(r-1)(D(n,m,n',m')-n'/2)}. \tag{35}$$

We first prove the proposition using Lemma 11.2.

Proof of Proposition 11.1. The structure of the proof is again similar to that of Proposition 9.2 though we consider combinations (n,k,m,n',k',m') in $(\mathcal{N} \oplus \mathbb{Z})^2$ that satisfy (R1)-(R4), as in the proof of Proposition 10.1. We fix $\mu \geq \nu' + 2d + 2$ and $\mu' > 6\beta/\delta + 2\beta$ and let $C(n,m,n',m')$ be that in Lemma 11.2 for such μ and μ'. For this time, we set

$$K_{n,k,m,n',k',m'} = C(n,m,n',m') \cdot 2^{-r_* \Delta(n,k,n',k')} \cdot \frac{w^{(\lambda)}(m')}{w^{(\lambda)}(m)}.$$

We use the following sublemma, whose proof is postponed for a while.

Sublemma 11.3. *There exists a constant $C_* > 0$ such that*

$$\sup_{(n,k,m)\in\mathcal{N}\oplus\mathbb{Z}}\left(\sum_{n',k',m':n,k,m} K_{n,k,m,n',k',m'}\right) < C_*\lambda\cdot 2^{-(\mu'-2\beta)\delta\lambda+4\beta\lambda}\|g\|_*$$

and

$$\sup_{(n',k',m')\in\mathcal{N}\oplus\mathbb{Z}}\left(\sum_{n,k,m:n',k',m'} K_{n,k,m,n',k',m'}\right) < C_*\lambda\cdot 2^{-(\mu'-2\beta)\delta\lambda+4\beta\lambda}\|g\|_*$$

where $\sum_{n',k',m':n,k,m}$ (resp. $\sum_{n,k,m:n',k',m'}$) *denotes the sum over* (n',k',m') (resp. (n,k,m)) *in* $\mathcal{N}\oplus\mathbb{Z}$ *such that* (n,k,m,n',k',m') *satisfies (R1)-(R4).*

Similarly to the proof of Proposition 9.2 and 10.1, we can deduce the following estimate from Lemma 11.2, by using Young and Schwarz inequality:

$$\sum_{\gamma':n',k',m'}\left\|{\sum_{\gamma:n,k,m;\gamma'}}^{\dagger} d^{\nu'}_{\gamma'}\mathcal{L}_{\gamma\gamma'}u_\gamma\right\|^2_{L^2} \le C_*\left|K_{n,k,m,n',k',m'}\frac{w^{(\lambda)}(m)}{w^{(\lambda)}(m')}\right|^2\|v_{n,k,m}\|^2_{L^2}$$

for $\mathbf{u} = (u_\gamma)_{\gamma\in\Gamma} \in \mathbf{B}^\beta_\nu$, where $v_{n,k,m}$ is defined by (24) and $\sum^\dagger_{\gamma:n,k,m;\gamma'}$ denotes the sum over $\gamma\in\Gamma$ such that $n(\gamma) = n$, $k(\gamma) = k$ and $m(\gamma) = m$ and that $(\gamma,\gamma')\in\mathcal{R}(4)$. But, once we have this estimate, we can proceed just as in the last part of the proof of Proposition 9.2, using Sublemma 11.3 instead of Sublemma 9.3, and conclude that

$$\|\mathcal{M}_4(\mathbf{u})\|^{(\lambda)}_{\beta,\nu'} \le C_*\|g\|_*\cdot\lambda\cdot 2^{-(\mu'-2\beta)\delta\lambda+4\beta\lambda}\cdot\|\mathbf{u}\|^{(\lambda)}_{\beta,\nu} \quad \text{for } \mathbf{u}\in\mathbf{B}^\beta_\nu.$$

Since we have $-(\mu'-2\beta)\delta\lambda+4\beta\lambda < -\beta\lambda$ from the choice of μ', this implies the conclusion of the proposition.

Below we complete the proof by proving Sublemma 11.3 and Lemma 11.2.

Proof of Sublemma 11.3. In the argument below, we consider combinations (n,k,m,n',k',m') in $(\mathcal{N}\oplus\mathbb{Z})^2$ that satisfy (R1)-(R4). First we restrict our attention to the case where (30) does *not* hold. Then $\max\{-m,m'\}\ge K/5$ from (R1) and (R5). Also we have

$$K_{n,k,m,n',k',m'} \le C(G,g)2^{-(r-1-2\beta)\max\{-m,m'\}-(r*-(r-1))\Delta(n,k,n',k')}$$

from (12) and (19). Taking (R5), (13) and (14) into consideration, we can get the inequalities in Sublemma 11.3 with the sum restricted to this case.

We next restrict our attention to the case where (30) holds. Then we have $\max\{n,n'\}\ge K$ from (R1) and $|n-n'|\le 1$ from (R8). Since

$$C(n,m,n',m') \le C_*\|g\|_* 2^{-\mu'\max\{-m,m'\}} + C(G,g)2^{-(r-1)(\max\{-m,m'\}+n/6)},$$

we see that $K_{n,k,m,n',k',m'}$ is bounded by

$$2^{4\beta\lambda - r_*\cdot\Delta(n,k,n',k')}\cdot$$
$$\left(C_*\|g\|_* 2^{-(\mu'-2\beta)\max\{-m,m'\}} + C(G,g)2^{-(r-1)K/6-(r-1-2\beta)\max\{-m,m'\}}\right).$$

Therefore, taking (R5) and (R7) into consideration, we obtain the required inequalities with the sum restricted to this case. $\square$

Proof of Lemma 11.2. Recall the vector $v_0 = \partial/\partial x_0$ and take unit vectors $v_1, v_2, \ldots, v_{2d}$ in E so that $\{v_j\}_{j=0}^{2d}$ is an orthonormal basis of E. Apply the formula (9) of integration by part along the set of vectors $\{v_j\}_{j=0}^{2d}$ for $(r-1)$ times to the integration with respect to the variable y in (15). Then, noting that (29) holds for all $y \in \operatorname{supp}\rho_{\gamma'}$, we can get the inequality in the lemma with (35) by straightforward estimate parallel to that in the proof of Lemma 7.5.

Remark 11.4. The result of integration by part above should appear more complicated than that in the proof of Lemma 7.5. But, since the constant $C(G,g)$ in (35) may depend on G and g, we can use rough estimates. As the result, we actually obtain the claim in the lemma with $d_{\gamma'}^{-2d-2}$ replaced by the indicator function of $Z(\gamma')$, which is a bit stronger than required.

We henceforth suppose that (30) hold. Then we have $|n(\gamma') - n(\gamma)| \le 1$ from (R8). Also it follows from the definition of $D(\gamma,\gamma')$ and (R5) that

$$D(\gamma,\gamma') \ge |m(\gamma')| + n(\gamma')/2 \ge n(\gamma')/2 \quad \text{for } (\gamma,\gamma') \in \mathcal{R}(4). \tag{36}$$

Since this implies that the diameter of $\operatorname{supp}\psi_{\gamma'}$ is not much larger than $2^{D(\gamma,\gamma')}$, we can construct a C^∞ partition of unity

$$\left\{ \phi_{\gamma\gamma'}^{(\ell)} : E^* \to [0,1] \;\middle|\; \ell = 0,1,2,\ldots \right\}$$

for each pair $(\gamma,\gamma') \in \mathcal{R}(4)$ so that the following conditions hold:

- (P1) $\operatorname{supp}\phi_{\gamma\gamma'}^{(0)}$ is contained in the $2^{D(\gamma,\gamma')-11}$-neighborhood of $\operatorname{supp}\psi_{\gamma'}$,
- (P2) for $\ell \ge 1$, the distance between $\operatorname{supp}\psi_{\gamma'}$ and $\operatorname{supp}\phi_{\gamma\gamma'}^{(\ell)}$ is bounded from below by $2^{D(\gamma,\gamma')+\ell-13}$, and
- (P3) the family of functions $\phi_{\gamma\gamma'}^{(\ell)}$ for $(\gamma,\gamma') \in \mathcal{R}(4)$ and $\ell \ge 0$ are uniformly bounded up to scaling in the sense that all the functions

$$\phi_{\gamma\gamma'}^{(\ell)} \circ A_{\gamma'} \circ J_{D(\gamma,\gamma')+\ell,0} : E \to [0,1]$$

are supported in a bounded subset in E and their C^s norms are uniformly bounded for each $s > 0$,

where A_γ and $J_{n,m}$ are those defined in Subsection 5.4. (We give one way of the construction in Remark 11.5 at the end of this proof.)

Using the partitions of unity as above, we decompose the kernel (7) as

$$\kappa_{\gamma\gamma'}(x',x) = (2\pi)^{-3(2d+1)} \sum_{\ell=0}^{\infty} \kappa^{(\ell)}_{\gamma\gamma'}(x',x)$$

where we define $\kappa^{(\ell)}_{\gamma\gamma'}(x',x)$ as the integral

$$\int e^{if(x',y',y,x;\xi',\xi,\eta)} \rho_{\gamma'}(y')\widetilde{\rho}_{\gamma'}(y)g(y)\psi_{\gamma'}(\xi)\phi^{(\ell)}_{\gamma\gamma'}(\xi')\widetilde{\psi}_{\gamma}(\eta)dydy'd\xi d\xi' d\eta \tag{37}$$

with setting

$$f(x',y',y,x;\xi',\xi,\eta) = \langle \xi, x'-y'\rangle + \langle \xi', y'-y\rangle + \langle \eta, G(y)-x\rangle$$

and

$$\widetilde{\rho}_{\gamma'}(y) = \chi\big(2^{n(\gamma')/3+1}\|y - z(\gamma')\|\big).$$

(Note that $\widetilde{\rho}_{\gamma'}\cdot\rho_{\gamma'} \equiv \rho_{\gamma'}$ and also that $(2\pi)^{-(2d+1)}\int e^{i\langle\xi',y-y'\rangle}d\xi' = \delta(y-y')$.)

We estimate $\kappa^{(\ell)}_{\gamma\gamma'}(x',x)$ by applying integration by part to the integral with respect to the variables y and y' in (37). To this end, we extend the formula (9) of integration by part to oscillatory integrals on $E \times E$ trivially. And we regard y and y' as the former and latter variable on $E\times E$ respectively.

In the case $\ell = 0$, we apply the formula of integration by part as follows:

(i) first, integrate by part along the single vector (v_0, v_0) for r_* times if $\Delta(\gamma,\gamma') > 0$, but do noting otherwise,

(ii) second, integrate by part along the set of vectors $\{(v_i, 0)\}_{i=0}^{2d}$ for $(r-1)$ times.

Note that, from Lemma 9.4 and (P1), we have

$$d(\Pi_{z(\gamma')}(\mathrm{supp}\,\phi^{(0)}_{\gamma\gamma'}), \Pi_{z(\gamma')}(DG^*_y(\mathrm{supp}\,\widetilde{\psi}_\gamma))) \geq 2^{D(\gamma,\gamma')-11}$$

for all $y \in \mathrm{supp}\,\widetilde{\rho}_{\gamma'}$. Using this and the condition (P3) in the result of integration by part as above and recalling Remark 11.4, we obtain

$$|\kappa^{(0)}_{\gamma\gamma'}(x',x)| \leq C(G,g)\cdot 2^{-r_*\cdot\Delta(\gamma,\gamma')-(r-1)(D(\gamma,\gamma')-n(\gamma')/3)}$$
$$\cdot \int \left(\int_{Z(\gamma')} b^{\mu}_{\gamma'}(x'-y') b^{\mu+2d+2}_{\gamma\gamma',0}(y'-y)dy' \right) b^{\mu}_{\gamma}(G(y)-x)dy$$

where (and also below) we set

$$b^{\mu+2d+2}_{\gamma\gamma',\ell}(x) = 2^{(2d+1)(D(\gamma,\gamma')+\ell)} \left\langle 2^{D(\gamma,\gamma')+\ell}\cdot x \right\rangle^{-\mu-2d-2} \quad \text{for } \ell \geq 0.$$

In the case $\ell \geq 1$, we apply the formula of integration by part as follows:

(i) first, integrate by part along the single vector (v_0, v_0) for r_* times if $\Delta(\gamma,\gamma') > 0$, but do nothing otherwise,

(ii) second, integrate by part along the set of vectors $\{(0, v_i)\}_{i=0}^{2d}$ for μ' times.

In the second step (ii), note that the differentiation along the vector $(0, v_i)$ does not create any term related to G or g. Using the condition (P2) and (P3) in the result of integration by part as above, we see that there exists a constant C_*, which is independent of G, g, λ and Λ, such that

$$|\kappa^{(\ell)}_{\gamma\gamma'}(x',x)| \le C_* \cdot \|g\|_* \cdot 2^{-r_*\cdot\Delta(\gamma,\gamma')-\mu'(D(\gamma,\gamma')+\ell-n(\gamma')/2)} \cdot \int \left(\int_{Z(\gamma')} b^{\mu}_{\gamma'}(x'-y') b^{\mu+2d+2}_{\gamma\gamma',\ell}(y'-y)dy'\right) b^{\mu}_{\gamma}(G(y)-x)dy.$$

By (36), there exists a constant $C_* > 0$ such that, for any $\ell \ge 0$,

$$\int_{Z(\gamma')} b^{\mu}_{\gamma'}(x'-y') b^{\mu+2d+2}_{\gamma\gamma',\ell}(y'-y)dy' \le C_* \cdot d^{-2d-2}_{\gamma'}(y) \cdot b^{\mu}_{\gamma'}(x'-y).$$

Therefore we can conclude the inequality in Lemma 11.2, by putting this inequality in the estimates on $\kappa^{(\ell)}_{\gamma\gamma'}(x',x)$ above and taking sum for $\ell \ge 0$. □

Remark 11.5. We can construct the partition of unity $\{\phi^{(\ell)}_{\gamma\gamma'}\}_{\ell\ge 0}$ with the properties (i), (ii) and (iii) in the proof above as follows: Let $K^{(\ell)}_{\gamma\gamma'}$ be the $2^{D(\gamma,\gamma')+\ell-12}$-neighborhood of supp $\psi_{\gamma'}$. Also we define $\phi_0 : E \to \mathbb{R}$ by

$$\phi_0(\eta) = \left(\int \chi(\|\xi\|)d\xi\right)^{-1} \cdot \chi(\|\eta\|)$$

where χ is the function taken in Subsection 5.1, so that it is supported on the disk with radius 5/3 and satisfies $\int \phi_0 d\eta = 1$. Then we set

$$H^{(\ell)}_{\gamma\gamma'}(\xi) = 2^{-(2d+1)(D(\gamma,\gamma')+\ell-13)} \int_{K^{(\ell)}_{\gamma\gamma'}} \phi_0\left(2^{-(D(\gamma,\gamma')+\ell-13)} \cdot \|\xi-\eta\|\right) d\eta$$

The function $H^{(\ell)}_{\gamma\gamma'}$ is supported on $K^{(\ell+1)}_{\gamma\gamma'}$ and satisfies $H^{(\ell)}_{\gamma\gamma'} \equiv 1$ on $K^{(\ell-1)}_{\gamma\gamma'}$. From (36), the required properties are fulfilled if we set

$$\phi^{(0)}_{\gamma\gamma'}(\xi) = H^{(0)}_{\gamma\gamma'}(\xi) \quad \text{and} \quad \phi^{(\ell)}_{\gamma\gamma'}(\xi) = H^{(\ell)}_{\gamma\gamma'}(\xi) - H^{(\ell-1)}_{\gamma\gamma'}(\xi) \quad \text{for } \ell \ge 1.$$

12. The central part of the operator $\mathcal{M}$

In this section, we consider the central part $\mathcal{M}_1$ defined in Subsection 8.1. Our goal is to prove the following proposition.

Proposition 12.1. *The formal definition of $\mathcal{M}_1$ in fact gives a bounded operator $\mathcal{M}_1 : \mathbf{B}^{\beta}_{\nu} \to \mathbf{B}^{\beta}_{\nu'}$ for any $\nu, \nu' \ge 2d+2$. Further, for the case $\nu = \nu' = \nu_*$, there exists a constant $C_* > 0$ such that we have*

$$\|\mathcal{M}_1(\mathbf{u})\|^{(\lambda)}_{\beta,\nu_*} \le C_* \cdot \|g\|_* \cdot 2^{-(1-\epsilon)\Lambda/2}\|\mathbf{u}\|^{(\lambda)}_{\beta,\nu_*} \quad \text{for } \mathbf{u} \in \mathbf{B}^{\beta}_{\nu_*}, \tag{38}$$

for $G \in \mathcal{H}(\Lambda,\lambda)$ and $g \in \mathscr{C}^r(V')$, provided $\Lambda \ge \Lambda_*$, $\lambda \ge \lambda_*$ and $\Lambda \ge d\lambda$.

Clearly we can conclude Theorem 7.2 from Proposition 8.1, 9.2, 10.1, 11.1 and Proposition 12.1 above, setting $\mathcal{K}(G,g) = \mathcal{M}_0$ and $\|\cdot\|^{(\lambda)} = \|\cdot\|^{(\lambda)}_{\beta,\nu_*}$.

12.1. **Reduction of the claim.** For integers $n, n' \geq 0$, we set

$$\mathcal{R}^{(n,n')}(1) = \{(\gamma, \gamma') \in \mathcal{R}(1) \mid n(\gamma) = n,\ n(\gamma') = n'\}$$

and let $\mathcal{M}_1^{(n,n')} : \mathbf{B}^\beta_\nu \to \mathbf{B}^\beta_{\nu'}$ be the operator defined formally by (17) with $\mathcal{R}(j)$ replaced by $\mathcal{R}^{(n,n')}(1)$. Then $\mathcal{M}_1$ is formally the sum of $\mathcal{M}_1^{(n,n')}$ for $(n, n') \in \mathbb{Z}_+ \times \mathbb{Z}_+$ such that $\max\{n, n'\} > K$ and $|n' - n| \leq 1$. From the definition of the norm $\|\cdot\|^{(\lambda)}_{\beta,\nu_*}$, Proposition 12.1 follows if we prove the claim (38) with $\mathcal{M}_1$ replaced by $\mathcal{M}_1^{(n,n')}$ and with the constant C_* independent of n and n'.

Let $\widetilde{\alpha}_0$ be the contact form defined by

$$\widetilde{\alpha}_0 = dx_0 + x^- \cdot dx^+.$$

It satisfies $H_0^*(\alpha_0) = \widetilde{\alpha}_0$ for the diffeomorphism $H_0 : E \to E$ defined by

$$H_0(x_0, x^+, x^-) = (x_0 + 2^{-1}x^+ \cdot x^-,\ 2^{-1/2}x^+,\ 2^{-1/2}x^-).$$

In the proof below, we regard the diffeomorphism $G : V' \to V$ in $\mathcal{H}(\lambda, \Lambda)$ as the composition of two contact diffeomorphisms

$$(V', \alpha_0) \xrightarrow{H_0^{-1}} (H_0^{-1}(V'), \widetilde{\alpha}_0) \xrightarrow{G\circ H_0} (E, \alpha_0).$$

Also we will introduce a Hilbert space $\widetilde{\mathbf{B}}$ and regard $\mathcal{M}_1^{(n,n')} : \mathbf{B}^\beta_\nu \to \mathbf{B}^\beta_{\nu'}$ as the composition of two operators $\mathcal{P}^{(n')}$ and $\mathcal{Q}^{(n)}$,

$$\mathbf{B}^\beta_\nu \xrightarrow{\mathcal{Q}^{(n)}} \widetilde{\mathbf{B}} \xrightarrow{\mathcal{P}^{(n')}} \mathbf{B}^\beta_{\nu'},$$

which are associated to the diffeomorphisms $G \circ H_0$ and H_0^{-1} respectively. As we will see in the next subsection, the operator $\mathcal{P}^{(n')}$ does nothing harmful and Proposition 12.1 is basically reduced to a claim on the operator $\mathcal{Q}^{(n)}$. The reason for taking this roundabout way is that we need to "straighten" the contact form α_0 along the subspace $E_0 \oplus E^+$ so that we can use the formula (9) of integration by part appropriately in the last part of the proof.

We define the transfer operators

$$P : C^r(H_0^{-1}(V')) \to C^r(V') \quad \text{and} \quad Q : C^r(V) \to C^r(H_0^{-1}(V'))$$

by $Pu = u \circ H_0^{-1}$ and $Qu = \hat{g} \cdot (u \circ \hat{G})$ respectively, where we set $\hat{g} = g \circ H_0$ and $\hat{G} = G \circ H_0$. Obviously we have $\mathcal{L} = P \circ Q$.

The definition of the Hilbert space $\widetilde{\mathbf{B}}$ is somewhat similar to that of $\mathbf{B}^\beta_\nu$. We consider $\Sigma = \mathcal{N} \oplus (\mathbb{Z}_+)$ as the index set instead of Γ. To refer the components of an element $\sigma = (n, k, \ell) \in \Sigma$, we set $n(\sigma) = n$, $k(\sigma) = k$ and $\ell(\sigma) = \ell$. For each $\sigma \in \Sigma$, we define the functions $\Psi_\sigma : E^* \to [0, 1]$ and $\widetilde{\Psi}_\sigma : E^* \to [0, 1]$ by

$$\Psi_\sigma(\xi) = \chi_{n(\sigma),k(\sigma)}(\xi) \cdot \chi_{\ell(\sigma)}(2^{-n(\sigma)/2-2\delta\lambda}\|\xi^-\|)$$

and

$$\widetilde{\Psi}_\sigma(\xi) = \widetilde{\chi}_{n(\sigma),k(\sigma)}(\xi) \cdot \widetilde{\chi}_{\ell(\sigma)}(2^{-n(\sigma)/2-2\delta\lambda}\|\xi^-\|)$$

respectively, where $\xi = (\xi_0, \xi^+, \xi^-)$ and the functions $\chi_{n,k}$, $\widetilde{\chi}_{n,k}$, χ_n and $\widetilde{\chi}_n$ are those defined in Section 5. By definition, the family $\{\Psi_\sigma\}_{\sigma\in\Sigma}$ is a partition of unity on E^* and we have $\Psi_\sigma \cdot \widetilde{\Psi}_\sigma \equiv \Psi_\sigma$ for each $\sigma \in \Sigma$. Note that the functions $\Psi_\sigma(\xi)$ and $\widetilde{\Psi}_\sigma(\xi)$ do not depend on the component ξ^+ and hence its inverse Fourier transform is not a function in the usual sense but the tensor product of the Dirac δ-function on E_+ at the origin and a rapidly decaying function on $E_0 \oplus E_-$. For $\mu \geq 2d+2$, there exists a constant $C_* > 0$ such that

$$|\Psi_\sigma(D)u(x)| = |\mathbb{F}^{-1}\Psi_\sigma * u(x)| \leq C_* \cdot |b_\sigma^\mu * |u|(x)|$$

where b_σ^μ is the finite measure on E defined by

$$b_\sigma^\mu(x) = \frac{2^{d(n(\sigma)/2+\ell(\sigma)+2\delta\lambda)+n(\sigma)/2}}{\langle 2^{n(\sigma)/2+\ell(\sigma)+2\delta\lambda}x^-\rangle^\mu \cdot \langle 2^{n(\sigma)/2}x_0\rangle^\mu} \cdot \delta(x^+) \tag{39}$$

for $x = (x_0, x^+, x^-)$. For $\sigma \in \Sigma$, we set[5]

$$\widetilde{w}(\sigma) = \begin{cases} (\delta\lambda)^{1/2}, & \text{if } \ell(\sigma) = 0; \\ 2^{-\Lambda-\ell(\sigma)}, & \text{if } \ell(\sigma) > 0. \end{cases}$$

Then we define the Hilbert space $\widetilde{\mathbf{B}}$ as the linear space

$$\widetilde{\mathbf{B}} = \left\{ (v_\sigma)_{\sigma\in\Sigma} \;\middle|\; v_\sigma \in L^2(E),\ \widetilde{\Psi}_\sigma(D)v_\sigma = v_\sigma,\ \sum_\sigma \widetilde{w}(\sigma)^2 \|v_\sigma\|_{L^2}^2 < \infty \right\}$$

equipped with the norm $\|\cdot\|_{\widetilde{\mathbf{B}}}$ defined by

$$\|\mathbf{v}\|_{\widetilde{\mathbf{B}}} = \left(\sum_{\sigma\in\Sigma} \widetilde{w}(\sigma)^2 \|v_\sigma\|_{L^2}^2 \right)^{1/2} \qquad \text{for } \mathbf{v} = (v_\sigma)_{\sigma\in\Sigma} \in \widetilde{\mathbf{B}}.$$

For $n \geq 0$, $\sigma \in \Sigma$ and $\gamma \in \Gamma$, we define the operator $\mathcal{P}_{\sigma\gamma}^{(n)} : L^2(E) \to L^2(E)$ and $\mathcal{Q}_{\gamma\sigma}^{(n)} : L^2(E) \to L^2(E)$ respectively by

$$\mathcal{P}_{\sigma\gamma}^{(n)}(v) = \begin{cases} p_\gamma(x, D)^*(P(\widetilde{\Psi}_\sigma(D)v)), & \text{if } |m(\gamma)| \leq \delta\lambda \text{ and } n(\gamma) = n; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathcal{Q}_{\gamma\sigma}^{(n)}(u) = \begin{cases} \Psi_\sigma(D)(Q(\widetilde{\psi}_\gamma(D)u)), & \text{if } |m(\gamma)| \leq \delta\lambda \text{ and } n(\gamma) = n; \\ 0, & \text{otherwise.} \end{cases}$$

We define the operators $\mathcal{P}^{(n)} : \widetilde{\mathbf{B}} \to \mathbf{B}_{\nu'}^\beta$ and $\mathcal{Q}^{(n)} : \mathbf{B}_\nu^\beta \to \widetilde{\mathbf{B}}$ formally by

$$\mathcal{P}^{(n)}((v_\sigma)_{\sigma\in\Sigma}) = \left(\sum_{\sigma\in\Sigma} \mathcal{P}_{\sigma\gamma}^{(n)}(v_\sigma) \right)_{\gamma\in\Gamma}$$

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[5]The factors $(\delta\lambda)^{1/2}$ and $2^{-\Lambda}$ in the definition of $\widetilde{w}(\sigma)$ is not essential at all. But we put those factors to make the statements simpler.

and

$$\mathcal{Q}^{(n)}((u_\gamma)_{\gamma\in\Gamma}) = \left(\sum_{\gamma\in\Gamma} \mathcal{Q}^{(n)}_{\gamma\sigma}(u_\gamma)\right)_{\sigma\in\Sigma}.$$

Clearly we have $\mathcal{M}_1^{(n,n')} = \mathcal{P}^{(n')} \circ \mathcal{Q}^{(n)}$ in the formal level. Therefore, in order to prove Proposition 12.1, it is enough to show

Proposition 12.2. *The formal definition of the operator $\mathcal{P}^{(n)}$ for $n \geq K$ in fact gives a bounded operator $\mathcal{P}^{(n)} : \widetilde{\mathbf{B}} \to \mathbf{B}^{\beta}_{\nu'}$ for each $\nu' \geq 2d+2$. Further, for $\nu' \geq 2d+2$, there exists a constant $C_* > 0$ such that we have*

$$\|\mathcal{P}^{(n)}(\mathbf{v})\|^{(\lambda)}_{\beta,\nu'} \leq C_* \|\mathbf{v}\|_{\widetilde{\mathbf{B}}} \quad \text{for all } \mathbf{v} \in \widetilde{\mathbf{B}} \text{ and } n \geq K,$$

whenever $\Lambda \geq \Lambda_*$ and $\lambda \geq \lambda_*$.

Proposition 12.3. *The formal definition of the operator $\mathcal{Q}^{(n)}$ in fact gives a bounded operator $\mathcal{Q}^{(n)} : \mathbf{B}^{\beta}_{\nu} \to \widetilde{\mathbf{B}}$ for each $\nu \geq 2d+2$. For $\nu \geq 2d+2$, the operator norms of $\mathcal{Q}^{(n)} : \mathbf{B}^{\beta}_{\nu} \to \widetilde{\mathbf{B}}$ for $n \geq K$ is uniformly bounded. Further, for the case $\nu = \nu' = \nu_*$, there exists a constant $C_* > 0$ such that*

$$\|\mathcal{Q}^{(n)}(\mathbf{u})\|_{\widetilde{\mathbf{B}}} \leq C_* \|g\|_* \cdot 2^{-(1-\epsilon)\Lambda/2} \|\mathbf{u}\|^{(\lambda)}_{\beta,\nu_*} \quad \text{for all } \mathbf{u} \in \mathbf{B}^{\beta}_{\nu_*} \text{ and } n \geq K$$

whenever $G : V' \to V$ belongs to $\mathcal{H}(\Lambda, \lambda)$, $g \in \mathscr{C}^r(V')$, $\Lambda \geq \Lambda_*$, $\lambda \geq \lambda_*$ and $\Lambda \geq d\lambda$.

In the following subsections, we prove Proposition 12.2 and 12.3. We henceforth consider a fixed $n \geq K$ and write $\mathcal{P}$, $\mathcal{P}_{\sigma\gamma}$, $\mathcal{Q}$ and $\mathcal{Q}_{\gamma\sigma}$ respectively for $\mathcal{P}^{(n)}$, $\mathcal{P}^{(n)}_{\sigma\gamma}$, $\mathcal{Q}^{(n)}$ and $\mathcal{Q}^{(n)}_{\gamma\sigma}$ for simplicity, though we keep paying attention to dependence of them on n. Notice that we will write C_*, $C(G)$ and $C(G,g)$ only for constants that do not depend on n.

12.2. **The operator $\mathcal{P}$.** In this subsection, we consider the operator $\mathcal{P} = \mathcal{P}^{(n)}$ and prove Proposition 12.2. The structure of the proof is similar to that of Proposition 9.2. Fix some integers $\mu \geq \nu' + 2d + 2$ and $\mu' > 2\Lambda/(\delta\lambda)$. For $\sigma \in \Sigma$ and $k \in \mathbb{Z}$ such that $(n,k) \in \mathcal{N}$, we set

$$K_{\sigma,k} = 2^{-\mu'(\Delta(n(\sigma),k(\sigma),n,k)+\delta\lambda+\ell(\sigma))} \cdot (1/\widetilde{w}(\sigma))$$

if $\ell(\sigma) > 0$ and $n/2 \leq n(\sigma)/2 + \ell(\sigma)$, and otherwise we set

$$K_{\sigma,k} = 2^{-\mu'\cdot\Delta(n(\sigma),k(\sigma),n,k)} \cdot (1/\widetilde{w}(\sigma)).$$

We use the following sublemma, whose proof is postponed for a while.

Sublemma 12.4. *There exists a constant $C_* > 0$ such that*

$$\sup_{\sigma} \left(\sum_{k:(n,k)\in\mathcal{N}} K_{\sigma,k}\right) \leq \frac{C_*}{\sqrt{\delta\lambda}} \quad \text{and} \quad \sup_{k:(n,k)\in\mathcal{N}} \left(\sum_{\sigma} K_{\sigma,k}\right) \leq \frac{C_*}{\sqrt{\delta\lambda}}.$$

Consider a pair $(\sigma,\gamma)\in\Sigma\times\Gamma$ such that $n(\gamma)=n$ and $|m(\gamma)|\le\delta\lambda$. We regard the operator $\mathcal{P}_{\sigma\gamma}$ as an integral operator

$$\mathcal{P}_{\sigma\gamma}u(x')=(2\pi)^{-2(2d+1)}\int\kappa_{\sigma\gamma}(x',x)u(x)dx$$

with the kernel

$$\kappa_{\sigma\gamma}(x',x)=\int e^{i\langle\xi,x'-y\rangle+i\langle\eta,H_0^{-1}(y)-x\rangle}\rho_\gamma(y)\psi_\gamma(\xi)\widetilde{\Psi}_\sigma(\eta)dyd\xi d\eta. \tag{40}$$

To apply the formula (9) of integration by part to this kernel, we prepare two estimates. The first is a simple one that

$$d\left(\pi_0^*((DH_0)_y^*(\operatorname{supp}\widetilde{\Psi}_\sigma)),\pi_0^*(\operatorname{supp}\psi_\gamma)\right)\ge 2^{\Delta(n(\sigma),k(\sigma),n(\gamma),k(\gamma))+n(\gamma)/2}$$

for all $y\in E$ when $\Delta(n(\sigma),k(\sigma),n(\gamma),k(\gamma))>0$. This follows immediately from the definitions. The second is that

$$d\left((DH_0^{-1})_y^*(\operatorname{supp}\widetilde{\Psi}_\sigma),\operatorname{supp}\psi_\gamma\right)\ge 2^{n(\sigma)/2+\delta\lambda+\ell(\sigma)}$$

for all $y\in E$ if $\ell(\sigma)>0$ and $n(\gamma)/2\le n(\sigma)/2+\ell(\sigma)$. We can get this estimate by an elementary geometric argument using $H_0^*(\alpha_0)=\widetilde{\alpha}_0$ and the fact $\delta\lambda\ge\delta\lambda_*>10$.

Recall the vector $v_0=\partial/\partial x_0$ and take unit vectors v_j, $1\le j\le 2d$, so that $\{v_j\}_{j=0}^{2d}$ is an orthonormal basis of E. First, to the integral with respect to y in (40), we apply the formula (9) of integration by part along the single vector v_0 for μ' times if $\Delta(n(\sigma),k(\sigma),n(\gamma),k(\gamma))>0$. Second, to the result of the previous step, we apply the formula (9) of integration by part along the set of vectors $\{v_j\}_{j=0}^{2d}$ for μ' times if $\ell(\sigma)>0$ and $n(\gamma)/2\le n(\sigma)/2+\ell(\sigma)$. Then, by the two estimates prepared above, it is not difficult to see that there exists a constant $C_*>0$ such that

$$|\kappa_{\sigma\gamma}(x',x)|\le C_*K_{\sigma,k(\gamma)}\cdot\widetilde{w}(\sigma)\int_{Z(\gamma)}b_\gamma^\mu(x'-y)b_\sigma^\mu(H_0(y)-x)dy.$$

Using this and Sublemma 12.4, we can proceed as in the proof of Proposition 9.2 and obtain

$$\begin{aligned}\left(\|\mathcal{P}^{(n)}(\mathbf{v})\|_{\beta,\nu'}^{(\lambda)}\right)^2&\le\sum_{k:(n,k)\in\mathcal{N}}\sum_{m:|m|\le\delta\lambda}\left\|\sum_{\sigma\in\Sigma}K_{\sigma,k}\cdot\widetilde{w}(\sigma)\cdot|b_\sigma^\mu*v_\sigma|\right\|_{L^2}^2\\&\le C_*\sum_{\sigma\in\Sigma}\widetilde{w}(\sigma)^2\cdot\|v_\sigma\|_{L^2}^2=C_*\|\mathbf{v}\|_{\widetilde{\mathbf{B}}}^2\end{aligned}$$

for $\mathbf{v}=(v_\sigma)_{\sigma\in\Sigma}\in\widetilde{\mathbf{B}}$. Now we finish the proof by proving Sublemma 12.4.

Proof of Sublemma 12.4. We prove the former inequality. The latter can be proved similarly. If $\ell(\sigma)=0$, the sum $\sum_{k:(n,k)\in\mathcal{N}}K_{\sigma,k}$ is bounded by

$$C_*(\delta\lambda)^{-1/2}\sum_{k:(n,k)\in\mathcal{N}}2^{-\mu'\cdot\Delta(n(\sigma),k(\sigma),n,k)}\le C_*(\delta\lambda)^{-1/2}\quad\text{from (13).}$$

If $\ell(\sigma) > 0$, the sum $\sum_{k:(n,k)\in\mathcal{N}} K_{\sigma,k}$ is bounded by

$$C_* 2^{\Lambda+\ell(\sigma)} \left(\sum_k^{*} 2^{-\mu'\cdot(\Delta(n(\sigma),k(\sigma),n,k)+\delta\lambda+\ell(\sigma))} + \sum_k^{**} 2^{-\mu'\cdot\Delta(n(\sigma),k(\sigma),n,k)} \right)$$

where $\sum_k^*$ (resp. $\sum_k^{**}$) denotes the sum over $k \in \mathbb{Z}$ such that $(n,k) \in \mathcal{N}$ and $n \le n(\sigma)/2 + \ell(\sigma)$ (resp. $n > n(\sigma)/2 + \ell(\sigma)$). Estimate the first sum by using (13) and (14), and the second sum by using (11) and the fact that $|n - n(\sigma)| > 2$ holds whenever $n/2 > n(\sigma)/2 + \ell(\sigma)$. Then we see that the sum $\sum_{k:(n,k)\in\mathcal{N}} K_{\sigma,k}$ is bounded by $C_* 2^{\Lambda-\mu'\cdot\delta\lambda}$ and hence by $C_*(\delta\lambda)^{-1/2}$, since we have $\mu' \cdot \delta\lambda > 2\Lambda > \Lambda + \lambda$ from the choice of μ'. □

12.3. **The operator $\mathcal{Q}$.** In the remaining part of this section, we consider the operator $\mathcal{Q} = \mathcal{Q}^{(n)}$ and prove Proposition 12.3. Consider $(\gamma,\sigma) \in \Gamma \times \Sigma$ such that $n(\gamma) = n$ and $|m(\gamma)| \le \delta\lambda$. We regard the operator $\mathcal{Q}_{\gamma\sigma}$ as an integral operator

$$\mathcal{Q}_{\gamma\sigma} u(x') = (2\pi)^{-2(2d+1)} \int \kappa_{\gamma\sigma}(x',x) u(x) dx$$

with the kernel

$$\kappa_{\gamma\sigma}(x',x) = \int e^{i\langle \xi, x'-y\rangle + i\langle \eta, \hat{G}(y)-x\rangle} \hat{g}(y) \Psi_\sigma(\xi) \widetilde{\psi}_\gamma(\eta) dy d\xi d\eta.$$

We can show the following estimate in the same way as Lemma 7.3 and 7.5.

Lemma 12.5. *For $\mu \ge 2d+2$, there exists a constant $C_* > 0$ such that*

$$|\kappa_{\gamma\sigma}(x',x)| \le C_* \cdot \|g\|_{L^\infty} \int b_\sigma^\mu(x'-y) \cdot b_\gamma^\mu(\hat{G}(y)-x) dy$$

for any $(\gamma,\sigma) \in \Gamma \times \Sigma$ and, further, that the left hand side is bounded by

$$C_* \cdot 2^{-r_*(\Delta(n(\gamma),k(\gamma),n(\sigma),k(\sigma))+n(\sigma)/2)} \|g\|_* \cdot \int b_\sigma^\mu(x'-y) b_\gamma^\mu(\hat{G}(y)-x) dy$$

if $\Delta(n(\gamma),k(\gamma),n(\sigma),k(\sigma)) > 0$.

Remark 12.6. Notice that we have the additional term $n(\sigma)/2$ in the second claim above compared with Lemma 7.5. This is because there is no longer the term $\rho_{\gamma'}$ which produced the factor $2^{n(\gamma)/2}$ for each differentiation.

Let $S = S(n)$ be the set of pairs $(\gamma,\sigma) \in \Gamma \times \Sigma$ such that

$$n(\gamma) = n, \quad |m(\gamma)| \le \delta\lambda, \quad \ell(\sigma) = 0 \quad \text{and} \quad \Delta(n(\gamma),k(\gamma),n(\sigma),k(\sigma)) = 0.$$

We define the operator $\widehat{\mathcal{Q}} = \widehat{\mathcal{Q}}^{(n)} : \mathbf{B}_\nu^\beta \to \widetilde{\mathbf{B}}$ formally by

$$\widehat{\mathcal{Q}}(\mathbf{u}) = \left(\sum_{\gamma:(\gamma,\sigma)\in S} \mathcal{Q}_{\gamma\sigma}(u_\gamma) \right)_{\sigma\in\Sigma} \qquad \text{for } \mathbf{u} = (u_\gamma)_{\gamma\in\Gamma} \in \mathbf{B}_\nu^\beta.$$

This is actually the main part of the operator $\mathcal{Q}$ and, in fact, we consider this part in the following two subsections. In the next lemma, we show that the remainder part $\mathcal{Q}-\widehat{\mathcal{Q}}:\mathbf{B}^{\beta}_{\nu}\to\widetilde{\mathbf{B}}$ of $\mathcal{Q}$, defined by

$$(\mathcal{Q}-\widehat{\mathcal{Q}})(\mathbf{u})=\left(\sum_{\gamma:(\gamma,\sigma)\notin S}\mathcal{Q}_{\gamma\sigma}(u_\gamma)\right)_{\sigma\in\Sigma}\qquad\text{for }\mathbf{u}=(u_\gamma)_{\gamma\in\Gamma}\in\mathbf{B}^{\beta}_{\nu},$$

does not do harm.

Lemma 12.7. *The formal definition of* $(\mathcal{Q}-\widehat{\mathcal{Q}})$ *above in fact gives a bounded operator* $(\mathcal{Q}-\widehat{\mathcal{Q}}):\mathbf{B}^{\beta}_{\nu}\to\widetilde{\mathbf{B}}$ *for any* $\nu\geq 2d+2$. *Further, for* $\nu\geq 2d+2$, *there exists a constant* $C_*>0$ *such that*

$$\|(\mathcal{Q}-\widehat{\mathcal{Q}})(\mathbf{u})\|_{\widetilde{\mathbf{B}}}\leq C_* 2^{-\Lambda/2}\|g\|_*\|\mathbf{u}\|^{(\lambda)}_{\beta,\nu}\quad\textit{for }\mathbf{u}\in\mathbf{B}^{\beta}_{\nu}$$

if $G:V'\to V$ *belongs to* $\mathcal{H}(\Lambda,\lambda)$, $g\in\mathscr{C}^r(V')$, $\Lambda\geq\Lambda_*$, $\lambda\geq\lambda_*$ *and* $\Lambda\geq d\lambda$.

Proof. For $\sigma\in\Sigma$ and $k\in\mathbb{Z}$ such that $(n,k)\in\mathcal{N}$, we set

$$K_{k,\sigma}=\begin{cases}2^{-r_*(\Delta(n,k,n(\sigma),k(\sigma))+n(\sigma)/2)}\|g\|_*\widetilde{w}(\sigma), & \text{if }\Delta(n,k,n(\sigma),k(\sigma))>0;\\ \|g\|_*\cdot\widetilde{w}(\sigma), & \text{if }\begin{bmatrix}\Delta(n,k,n(\sigma),k(\sigma))=0\\ \text{and }\ell(\sigma)>0\end{bmatrix};\\ 0, & \text{otherwise.}\end{cases}$$

Then, recalling the assumption that $n\geq K$, it is not difficult to check that

$$\sup_{\sigma\in\Sigma}\left(\sum_{k:(n,k)\in\mathcal{N}}K_{k,\sigma}\right)\leq C_*2^{-\Lambda}\|g\|_*,\qquad\sup_{k:(n,k)\in\mathcal{N}}\left(\sum_{\sigma}K_{k,\sigma}\right)\leq C_*2^{-\Lambda}\|g\|_*.$$

By Lemma 12.5 and Young inequality, we have

$$\|(\mathcal{Q}-\widehat{\mathcal{Q}})(\mathbf{u})\|^2_{\widetilde{\mathbf{B}}}\leq C_*\sum_{\sigma}\left\|\sum_{m:|m|\leq\delta\lambda}\sum_{k:(n,k)\in\mathcal{N}}K_{k,\sigma}\cdot b^{\mu}_{n,m}*\left(\sum_{\gamma:n,k,m}|u_\gamma|\right)\right\|^2_{L^2}$$

for $\mathbf{u}=(u_\gamma)_{\gamma\in\Gamma}\in\mathbf{B}^{\beta}_{\nu}$, where $\sum_{\gamma:n,k,m}$ denotes the sum over $\gamma\in\Gamma$ such that $n(\gamma)=n$, $m(\gamma)=m$ and $k(\gamma)=k$. By Schwarz inequality, (25) and the inequalities above on the sums of $K_{k,\sigma}$, we obtain that

$$\begin{aligned}&\|(\mathcal{Q}-\widehat{\mathcal{Q}})(\mathbf{u})\|^2_{\widetilde{\mathbf{B}}}\\ &\quad\leq C_*\delta\lambda\cdot 2^{-\Lambda}\|g\|_*\cdot\sum_{\sigma}\sum_{m:|m|\leq\delta\lambda}\sum_{k:(n,k)\in\mathcal{N}}K_{k,\sigma}\sum_{\gamma:n,k,m}\|d^{\nu}_{\gamma}u_\gamma\|^2_{L^2}\\ &\quad\leq C_*(\delta\lambda)^2\cdot 2^{-2\Lambda}\|g\|^2_*\cdot\left(\|\mathbf{u}\|^{(\lambda)}_{\beta,\nu}\right)^2.\end{aligned}$$

From the assumption $\Lambda\geq d\lambda$, this implies the conclusion of the lemma. □

12.4. **The operator $\widehat{\mathcal{Q}}$.** In this subsection and the next, we consider the operator $\widehat{\mathcal{Q}} = \widehat{\mathcal{Q}}^{(n)} : \mathbf{B}^{\beta}_{\nu} \to \widetilde{\mathbf{B}}$. Using Lemma 12.5, it is easy to check that the formal definition of $\widehat{\mathcal{Q}}$ gives a bounded operator $\widehat{\mathcal{Q}} : \mathbf{B}^{\beta}_{\nu} \to \widetilde{\mathbf{B}}$ and the operator norm is bounded by $C_*(\delta\lambda)\|g\|_*$. This and Lemma 12.7 imply the former statement of Proposition 12.3 on boundedness of $\mathcal{Q}$. To prove the latter statement, we need more precise estimates.

Lemma 12.8. *If $(\gamma,\sigma) \in S$ and if $u \in L^2(E)$ satisfies $\widetilde{\psi}_\gamma(D)u = u$ and $\|d_\gamma^{\nu_*} u\|_{L^2} < \infty$, we have $\|\mathcal{Q}_{\gamma\sigma}(u)\|_{L^2} \le C_* 2^{-\Lambda/2 + d\delta\lambda}\|g\|_{L^\infty}\|d_\gamma^{\nu_*} u\|_{L^2}$.*

Proof. By using Schwarz inequality and Young inequality, we have

$$\begin{aligned}\|\mathcal{Q}_{\gamma\sigma}(u)\|_{L^2}^2 &\le C_*\|g\|_{L^\infty}^2 \left\| |\mathbb{F}^{-1}\Psi_\sigma| * \left|(d_\gamma^{\nu_*}u)^2 \circ \hat{G}\right| \cdot |\mathbb{F}^{-1}\Psi_\sigma| * \left|d_\gamma^{-2\nu_*} \circ \hat{G}\right|\right\|_{L^1} \\ &\le C_*\|g\|_{L^\infty}^2 \|d_\gamma^{\nu_*}u\|_{L^2}^2 \left\| |\mathbb{F}^{-1}\Psi_\sigma| * \left|d_\gamma^{-2\nu_*} \circ \hat{G}\right|\right\|_{L^\infty}.\end{aligned}$$

For the last factor, we can see, from (39) and the definition of $\mathcal{H}(\lambda,\Lambda)$, that

$$\left\| |\mathbb{F}^{-1}\Psi_\sigma| * \left|d_\gamma^{-2\nu_*} \circ \hat{G}\right|\right\|_{L^\infty} < C_* 2^{-\Lambda + 2d\delta\lambda}.$$

We therefore obtain the estimate in the lemma. □

The next lemma is the core of our argument on the central part.

Lemma 12.9. *There exists a constant $C_* > 0$ such that*

$$|\langle \mathcal{Q}_{\gamma\sigma}(u), \mathcal{Q}_{\gamma'\sigma}(u')\rangle_{L^2}| \le C_* \cdot \frac{2^{-\Lambda+2d\delta\lambda} \cdot \|g\|_{L^\infty}^2 \cdot \|d_\gamma^{\nu_*}u\|_{L^2} \cdot \|d_{\gamma'}^{\nu_*}u'\|_{L^2}}{\langle 2^{n/2-2\delta\lambda}\|z(\gamma) - z(\gamma')\|\rangle^{2d+2}}$$

for $(\gamma,\sigma), (\gamma',\sigma) \in S$ and $u, u' \in L^2(E)$ satisfying $\widetilde{\psi}_\gamma(D)u = u$, $\widetilde{\psi}_{\gamma'}(D)u' = u'$, $\|d_\gamma^{\nu_*}u\|_{L^2} < \infty$ and $\|d_{\gamma'}^{\nu_*}u'\|_{L^2} < \infty$.

Recalling the definition of S and Remark 7.4, we see that this implies

$$\begin{aligned}\|\widehat{\mathcal{Q}}(\mathbf{u})\|_{\widetilde{\mathbf{B}}}^2 &\le \widetilde{w}(0)^2 \cdot \sum_{\sigma\in\Sigma:\ell(\sigma)=0}\ \sum_{\gamma:(\gamma,\sigma)\in S}\ \sum_{\gamma':(\gamma',\sigma)\in S} |\langle \mathcal{Q}_{\gamma\sigma}(u_\gamma), \mathcal{Q}_{\gamma'\sigma}(u_{\gamma'})\rangle_{L^2}| \\ &\le C_* \cdot (\delta\lambda)^3 \cdot 2^{-\Lambda + (6d+4)\delta\lambda}\|g\|_{L^\infty}^2 \cdot (\|\mathbf{u}\|_{\beta,\nu_*}^{(\lambda)})^2\end{aligned}$$

for $\mathbf{u} = (u_\gamma)_{\gamma\in\Gamma} \in \mathbf{B}^{\beta}_{\nu_*}$ and hence the operator norm of $\widehat{\mathcal{Q}} : \mathbf{B}^{\beta}_{\nu_*} \to \widetilde{\mathbf{B}}$ is bounded by $C_* 2^{(1-\epsilon)\Lambda/2}$ from the choice of δ. Therefore the latter claim of Proposition 12.3 follows from Lemma 12.9 and Lemma 12.7.

We prove Lemma 12.9 in the remaining part of this subsection and in the next subsection. We consider $(\gamma,\sigma), (\gamma',\sigma) \in S$ and $u, u' \in L^2(E)$ satisfying the conditions in Lemma 12.9 and prove the conclusion of Lemma 12.9 in each of the following four cases separately:

(i) $\|z(\gamma) - z(\gamma')\| \le 2^{-n/2+2\delta\lambda}$,
(ii) $\|z(\gamma) - z(\gamma')\| \ge 2^{(-1/2+\tau)n}$ with $\tau = 1/(5(d+1))$,
(iii) neither (i) nor (ii), but $\|\pi_-(z(\gamma) - z(\gamma'))\| \le \|z(\gamma) - z(\gamma')\|/10$,
(iv) neither (i) nor (ii), but $\|\pi_-(z(\gamma) - z(\gamma'))\| > \|z(\gamma) - z(\gamma')\|/10$.

In the case (i), Lemma 12.9 is an immediate consequence of Lemma 12.8. The case (iv) is the most important case and considered in the next subsection. Below we deal with the other cases.

Proof of Lemma 12.9 in the case (ii) *and* (iii). Let us consider the case (iii) first. We first prove

Sublemma 12.10. *There exists a constant $C_* > 0$ such that, for each point $y \in E$, either of the following two conditions holds: the condition that*

$$\frac{\mathbf{1}_{\operatorname{supp} \hat{g}}(y-z)}{\langle 2^{n/2} z\rangle d_\gamma(\hat{G}(y-z))} \le \frac{C_*}{2^{n/2}\|z(\gamma')-z(\gamma)\|} \qquad \forall z \in E_0 \oplus E_-, \tag{41}$$

or the same condition with γ and γ' exchanged.

Proof. If $\|z\| \ge \|z(\gamma)-z(\gamma')\|/100$, both of the conditions hold with $C_* = 100$ obviously. Otherwise, since we are considering the case (iii), we can show the claim of the lemma by a simple geometric argument using the assumptions that $G \in \mathcal{H}(\lambda, \Lambda)$ and that $n \ge K$. □

Let Y be the set of points $y \in E$ for which (41) holds. Then, for $y \in Y$, $\mathcal{Q}_{\gamma\sigma}(u)(y) = \mathbb{F}^{-1}\Psi_\sigma * (\hat{g}\cdot(u\circ\hat{G}))(y)$ is bounded in absolute value by

$$C_*(2^{n/2}\|z(\gamma')-z(\gamma)\|)^{-(2d+2)}\cdot\|g\|_{L^\infty}\cdot$$
$$\int \left(\mathbb{F}^{-1}\Psi_\sigma(z)\cdot\langle 2^{n/2}z\rangle^{2d+2}\right)\cdot(d_\gamma^{2d+2}\circ\hat{G})(y-z)\cdot u\circ\hat{G}(y-z)dz.$$

Applying the argument in the proof of Lemma 12.8 to the integral above with slight modification, we obtain that

$$\|\mathcal{Q}_{\gamma\sigma}(u)\cdot\mathbf{1}_Y\|_{L^2}^2 \le C_* 2^{-\Lambda+d\delta\lambda}\|g\|_{L^\infty}^2(2^{n/2}\|z(\gamma')-z(\gamma)\|)^{-2(2d+2)}\|d_\gamma^{\nu_*}u\|_{L^2}^2.$$

Exchanging γ and γ', we obtain the same estimate for $\|\mathcal{Q}_{\gamma'\sigma}(u')\cdot\mathbf{1}_{E\setminus Y}\|_{L^2}^2$. Since $|\langle\mathcal{Q}_{\gamma\sigma}(u),\mathcal{Q}_{\gamma'\sigma}(u')\rangle_{L^2}|$ is bounded by

$$\|\mathcal{Q}_{\gamma\sigma}(u)\cdot\mathbf{1}_E\|_{L^2}\cdot\|\mathcal{Q}_{\gamma'\sigma}(u')\|_{L^2}+\|\mathcal{Q}_{\gamma'\sigma}(u')\cdot\mathbf{1}_{E\setminus Y}\|_{L^2}\cdot\|\mathcal{Q}_{\gamma\sigma}(u)\|_{L^2},$$

we get the conclusion of Lemma 12.9 from these estimates and Lemma 12.8.

We next consider the case (ii). Note that we could show the claim of Sublemma 12.10 in the case (ii) easily *if we allowed the constant $C_* > 0$ in it to depend on G*. Thus, following the argument above for the case (iii) and replacing $2d+2$ by $2d+3$ there, we reach the estimate

$$|\langle\mathcal{Q}_{\gamma\sigma}(u),\mathcal{Q}_{\gamma'\sigma}(u')\rangle_{L^2}| \le C(G)\frac{2^{-\Lambda+2d\delta\lambda}\|g\|_{L^\infty}^2\cdot\|d_\gamma^{\nu_*}u\|_{L^2}\cdot\|d_{\gamma'}^{\nu_*}u'\|_{L^2}}{\langle 2^{n/2}\|z(\gamma)-z(\gamma')\|\rangle^{2d+3}}$$

with $C(G)$ a constant which depends on the diffeomorphism G. But this implies the lemma because $C(G)/\langle 2^{n/2}\|z(\gamma)-z(\gamma')\|\rangle < C(G)2^{-\tau n} < 1$ in the case (ii), provided that we take large K depending on G. □

12.5. **The main part of the proof of Lemma 12.9.** In this subsection, we prove Lemma 12.9 in the case (iv). This complete the proof of Proposition 12.1 and hence that of the main theorem.

If either $z(\gamma)$ or $z(\gamma')$ is not contained in the image $G(V')$ of $\hat{G}$, we have $d_\gamma(y) \geq C(G,g)2^{n/2}$ and $d_{\gamma'}(y) \geq C(G,g)2^{n/2}$ for all $y \in \operatorname{supp} \hat{g}$ and hence we can prove the conclusion of Lemma 12.9 easily, taking large K according to G and g. Therefore we henceforth suppose that $z(\gamma)$ and $z(\gamma')$ are contained in $G(V')$ and let $y(\gamma)$ and $y(\gamma')$ be the unique points in V' such that $\hat{G}(y(\gamma)) = z(\gamma)$ and $\hat{G}(y(\gamma')) = z(\gamma')$ respectively.

In order to cut off the tail part of $\langle \mathcal{Q}_{\gamma\sigma}(u), \mathcal{Q}_{\gamma'\sigma}(u')\rangle_{L^2}$, we define the C^∞ functions $h, h' : E \to [0,1]$ by

$$h(y) = \chi\left(\frac{20\|\pi_-(D\hat{G}_{z(\gamma)}(y - y(\gamma)))\|}{\|\pi_-(z(\gamma) - z(\gamma'))\|}\right) \cdot \chi(2^{n/3} \cdot \|y - y(\gamma)\|)$$

and $h'(y) = h(y - y(\gamma) + y(\gamma'))$, where χ is the function defined in the beginning of Section 5. Notice that the supports of h and h' are contained in the disk with center at $y(\gamma)$ and radius $2^{-n/3+1}$ and that $\hat{G}$ is well approximated by its linearization at $y(\gamma)$ on that disk up to the error term bounded by $C(G)(2^{-n/3})^2 \ll 2^{-n/2}$. In particular, we have

$$d_\gamma^{-1} \circ \hat{G}(y) \leq C_* 2^{-n/2} \|\pi_-(z(\gamma) - z(\gamma'))\|^{-1} \quad \text{for } y \in \operatorname{supp}(1-h), \text{ and}$$
$$d_{\gamma'}^{-1} \circ \hat{G}(y) \leq C_* 2^{-n/2} \|\pi_-(z(\gamma) - z(\gamma'))\|^{-1} \quad \text{for } y \in \operatorname{supp}(1-h').$$

Let us set $v = \Psi_\sigma(D)(h \cdot Q(u))$ and $v' = \Psi_\sigma(D)(h' \cdot Q(u'))$. Then, applying the same argument as in the proof of Lemma 12.8, we see that $\|\mathcal{Q}_{\gamma\sigma}(u) - v\|_{L^2}^2 = \|\Psi_\sigma(D)((1-h) \cdot Q(u))\|_{L^2}^2$ is bounded by

$$\|g\|_{L^\infty}^2 \cdot \|d_\gamma^{\nu_*} u\|_{L^2}^2 \cdot \left\| |\mathbb{F}^{-1}\Psi_\sigma| * \left| (1-h) \cdot d_\gamma^{-2\nu_*} \circ \hat{G} \right| \right\|_{L^\infty}.$$

From the estimate on $d_\gamma^{-1} \circ \hat{G}$ above, the last factor above is bounded by

$$C_* \cdot 2^{-\Lambda + 2d\delta\lambda} \cdot \langle 2^{n/2} \|\pi_-(z(\gamma) - z(\gamma'))\| \rangle^{-2\nu_* + d + 2}.$$

Hence we obtain

$$\|\mathcal{Q}_{\gamma\sigma}(u) - v\|_{L^2}^2 \leq C_* \cdot \frac{2^{-\Lambda + 2d\delta\lambda} \cdot \|g\|_{L^\infty}^2 \cdot \|d_\gamma^{\nu_*} u\|_{L^2}^2}{\langle 2^{n/2} \|z(\gamma) - z(\gamma')\| \rangle^{2\nu_* - d - 2}}.$$

Similarly we obtain the parallel estimate for $\|\mathcal{Q}_{\gamma'\sigma}(u') - v'\|_{L^2}^2$. Therefore, by Lemma 12.8 and the choice of ν_*, we have

$$|\langle \mathcal{Q}_{\gamma\sigma}(u), \mathcal{Q}_{\gamma'\sigma}(u')\rangle_{L^2} - \langle v, v'\rangle_{L^2}| \leq C_* \cdot \frac{2^{-\Lambda + 2d\delta\lambda} \|g\|_{L^\infty}^2 \|d_\gamma^{\nu_*} u\|_{L^2} \|d_{\gamma'}^{\nu_*} u'\|_{L^2}}{\langle 2^{n/2} \|z(\gamma) - z(\gamma')\| \rangle^{2d+2}}.$$

Now it is left to show that

$$|\langle v, v'\rangle_{L^2}| \leq C_* \frac{2^{-\Lambda + 2d\delta\lambda} \|g\|_{L^\infty}^2 \|d_\gamma^{\nu_*} u\|_{L^2} \|d_{\gamma'}^{\nu_*} u'\|_{L^2}}{\langle 2^{n/2 - 2\delta\lambda} \|z(\gamma) - z(\gamma')\| \rangle^{2d+2}}. \tag{42}$$

Set $f(y,z,\xi,\xi') = \langle \xi', \hat{G}(y+z)\rangle - \langle \xi, \hat{G}(y)\rangle$ and

$$\mathcal{K}(z) = \mathbb{F}\Psi_\sigma * \mathbb{F}\Psi_\sigma(z) = \int \mathbb{F}\Psi_\sigma(z')\cdot \mathbb{F}\Psi_\sigma(z-z')dz'.$$

Then we can rewrite $\langle v, v'\rangle_{L^2}$ as

$$\langle v, v'\rangle_{L^2} = (2\pi)^{-2(2d+1)} \int \mathcal{K}(z)\left(\int S(x,x';z)\cdot \overline{u(x)}\cdot u'(x')dxdx'\right)dz,$$

writing $S(x,x';z)$ for the integral

$$\int e^{-i\langle \xi, x\rangle + i\langle \xi', x'\rangle - if(y,z,\xi,\xi')}\widetilde{\psi}_\gamma(\xi)\widetilde{\psi}_{\gamma'}(\xi')\hat{g}(y)\hat{g}(y+z)h(y)h'(y+z)dyd\xi d\xi'.$$

Note that $\mathcal{K}(z)$ is the tensor product of the Dirac δ-function on E_+ at the origin and a rapidly decaying function on $E_0\oplus E_-$.

We are going to apply the formula (9) of integration by part to the integral with respect to the variable y in $S(x,x';z)$ above. To this end, we set up a unit vector $w\in E$ along which we integrate by part. Recall that we have

$$d\alpha_0 = 2\cdot dx^-\wedge dx^+ = 2\sum_{i=1}^d dx_i^-\wedge dx_i^+.$$

We define w as the unique unit vector such that $D\hat{G}_{y(\gamma)}(w)\in E_0\oplus E_+$, that

$$\langle \widetilde{\alpha}_0(y(\gamma)), w\rangle = \langle \alpha_0(z(\gamma)), D\hat{G}_{y(\gamma)}(w)\rangle = 0$$

and that

$$d\alpha_0(D\hat{G}_{y(\gamma)}(w), \pi_-(z(\gamma')-z(\gamma))) \\ = 2\|\pi_-(z(\gamma')-z(\gamma))\|\|\pi_+(D\hat{G}_{y(\gamma)}(w))\|.$$

We write D_w for the directional derivative along w. Then it holds

$$D_w f(y,z,\xi,\xi') = \langle \xi', D\hat{G}_{y+z}(w)\rangle - \langle \xi, D\hat{G}_y(w)\rangle. \tag{43}$$

The next sublemma tells that the term $e^{-if(y,z,\xi,\xi')}$ in $S(x,x';z)$ as a function of y oscillates very fast in the direction of w.

Sublemma 12.11. *If $y+z\in \operatorname{supp} h'$ for $y\in\operatorname{supp} h$ and $z\in E_0\oplus E_-$, and if $\xi\in\operatorname{supp}\widetilde{\psi}_\gamma$ and $\xi'\in\operatorname{supp}\widetilde{\psi}_{\gamma'}$, we have*

$$|D_w f(y,z,\xi,\xi')| \ge 2^{n-10}\cdot \|\pi_-(z(\gamma')-z(\gamma))\|\cdot\|\pi_+(D\hat{G}_{y(\gamma)}(w)))\|.$$

We postpone the proof of this sublemma for a while. Note that, under the same assumption as in the sublemma above, we have

$$|D_w^k f(y,z,\xi,\xi')| \le C(G)\cdot 2^n\cdot \|z(\gamma')-z(\gamma)\| \quad \text{for } k=1,2 \tag{44}$$

for some constant $C(G)$ that depend on G. Also note that we have

$$\|D_w h\|_{L^\infty} \le C_* 2^{n/3} \quad \text{and} \quad \|D_w h'\|_{L^\infty}\le C_* 2^{n/3}. \tag{45}$$

Now we apply the formula (9) of integration by part along the single vector w once to the integration with respect to y in the integral $S(x,x';z)$. Then the result should be of the form

$$\int e^{-i\langle \xi,x\rangle+i\langle \xi',x'\rangle-if(y,z,\xi,\xi')}\widetilde{\psi}_\gamma(\xi)\widetilde{\psi}_\gamma(\xi')R(y,z;x,x';\xi,\xi')dyd\xi d\xi'.$$

By using Sublemma 12.11, (44) and (45), there exists a constant $C_{\alpha,\beta}(G,g)$ for multi-indices α and β, which may depend on G, g and λ, such that

$$\|\partial_\xi^\alpha\partial_{\xi'}^\beta R\|_{L^\infty}\le \frac{C_{\alpha,\beta}(G,g)\cdot 2^{-(|\alpha|+|\beta|)n/2}}{2^n\|z(\gamma)-z(\gamma')\|}.$$

Therefore we have that

$$|S(x,x';z)|\le C(G,g)\int \frac{|\mathcal{K}(z)|b_{n,0}^{2d+2}(\hat{G}(y)-x)b_{n,0}^{2d+2}(\hat{G}(y+z)-x')}{2^n\|z(\gamma)-z(\gamma')\|}dzdy$$

where $b_{n,m}^{\mu}$ is the function defined in (2). By Young inequality, we obtain

$$|\langle v,v'\rangle_{L^2}|\le C(G,g)\cdot\frac{\|d_\gamma^{\nu_*}u\|_{L^2}\|d_{\gamma'}^{\nu_*}u'\|_{L^2}}{2^n\|z(\gamma)-z(\gamma')\|}.$$

This implies (42), since we have $(2d+2)\tau<1/2$ from the choice of τ and

$$(2^{n/2}\|z(\gamma)-z(\gamma')\|)^{2d+2}\le 2^{(2d+2)\cdot\tau n}\le 2^{((2d+2)\cdot\tau-1/2)n}\cdot 2^n\|z(\gamma)-z(\gamma')\|.$$

(Recall that $n\ge K$ and that we may take large K depending on G and g.)

Finally we complete the proof by proving Sublemma 12.11.

Proof of Sublemma 12.11. Recall that the supports of h and h' are contained in the disk with center at $y(\gamma)$ and radius $2^{-n/3+1}$ and that $\hat{G}$ is well approximated by its linearization at $y(\gamma)$ on that disk. From the assumption that y and $y+z$ belong to $\operatorname{supp} h$ and $\operatorname{supp} h'$ respectively, we see that

$$\|\pi_-(D\hat{G}_{y(\gamma)}(z)-z(\gamma')-z(\gamma))\|<\|\pi_-(z(\gamma')-z(\gamma))\|/4.$$

From the choice of the vector w, we see that

$$\begin{aligned}|\langle\alpha_0(\hat{G}(y+z)),&D\hat{G}_{y+z}(w)\rangle-\langle\alpha_0(\hat{G}(y)),D\hat{G}_y(w)\rangle|\\ &=|\langle\widetilde{\alpha}_0(y+z),w\rangle-\langle\widetilde{\alpha}_0(y),w\rangle|\ge(9/10)|d\widetilde{\alpha}_0(z,w)|\\ &\ge|d\alpha_0(\pi_-(z(\gamma')-z(\gamma)),D\hat{G}_{y(\gamma)}(w))|/2\\ &=\|\pi_-(z(\gamma')-z(\gamma))\|\|\pi_+(D\hat{G}_{y(\gamma)}(w))\|.\end{aligned}$$

Since $n(\gamma)=n(\gamma')=n$ and

$$\Delta(n(\gamma),k(\gamma),n(\sigma),k(\sigma))=\Delta(n(\gamma'),k(\gamma'),n(\sigma),k(\sigma))=0$$

from the definition of S, we have

$$2^{n-2}\le|\xi_0|\le 2^{n+2},\quad 2^{n-2}\le|\xi_0'|\le 2^{n+2}\quad\text{and}\quad|\xi_0-\xi_0'|\le 2^{n/2+5}$$

for $\xi_0=\pi_0^*(\xi)$ and $\xi_0'=\pi_0^*(\xi')$. Therefore the lemma follows if we show

$$\langle\xi_0\cdot\alpha_0(\hat{G}(y+z))-\xi',D\hat{G}_{y+z}(w)\rangle\le|\xi_0|\|\pi_-(z(\gamma)-z(\gamma'))\|\|D\hat{G}_{y(\gamma)}(w)\|/3$$

and

$$\langle \xi_0 \cdot \alpha_0(\hat{G}(y)) - \xi, D\hat{G}_y(w)\rangle \le |\xi_0| \|\pi_-(z(\gamma) - z(\gamma'))\| \|D\hat{G}_{y(\gamma)}(w)\|/3.$$

But we can prove these by a straightforward estimate. Below we prove the former inequality and the latter can be proved similarly.

Since we have $y + z \in \operatorname{supp} h'$, $\xi' \in \operatorname{supp} \widetilde{\psi}_{\gamma'}$ and $|m(\gamma')| \le \delta\lambda$, it holds

$$\begin{aligned}
&\|\pi_{+,0}(D\hat{G}_{y+z}(w))\| \le 2\|D\hat{G}_{y(\gamma)}(w)\|, \quad \text{and} \\
&\|\pi^*_{+,0}(\xi_0 \cdot \alpha_0(\hat{G}(y+z)) - \xi')\| \\
&\qquad \le |\xi_0| \cdot \|\pi^*_+(\alpha_0(\hat{G}(y+z)) - \alpha_0(z(\gamma')))\| + |\xi_0 - \xi'_0| \cdot \|\pi^*_+(\alpha_0(z(\gamma')))\| \\
&\qquad\qquad + \|\pi^*_+(\xi'_0 \cdot \alpha_0(z(\gamma'))) - \xi)\| \\
&\qquad \le |\xi_0| \|\pi_-(\hat{G}(y+z) - \hat{G}(z(\gamma')))\| + 2^{n/2+6} + 2^{n/2+\delta\lambda+5} \\
&\qquad \le |\xi_0| \|\pi_-(z(\gamma) - z(\gamma'))\|/10
\end{aligned}$$

where, in the last inequality, we used the facts that $\delta\lambda \ge \delta\lambda_* \ge 10$ and that

$$\|\pi_-(z(\gamma) - z(\gamma'))\| \ge \|z(\gamma) - z(\gamma')\|/10 > 2^{-n/2+2\delta\lambda-4}.$$

Also we have, by rough estimate, that

$$\begin{aligned}
&\|\pi_-(D\hat{G}_{y+z}(w))\| \le C(G)\|(y+z) - y(\gamma)\| \le C(G)2^{-n/3} \quad \text{and} \\
&\|\pi^*_-(\xi_0 \cdot \alpha_0(\hat{G}(y+z)) - \xi')\| \le C(G)2^{(2/3)n}.
\end{aligned}$$

Clearly these imply the required inequality. □

Appendix A. Proof of Lemma 6.2

Let $p_n(\xi) = \chi_n(|\xi|)$ and $\tilde{p}_n(\xi) = \widetilde{\chi}_n(|\xi|)$ for $n \ge 0$, where χ_n and $\widetilde{\chi}_n$ are those defined in Subsection 5.2. For $u \in C^\infty(\mathbb{D})$, we define $u_\gamma = p_\gamma(x, D)^* u$ for $\gamma \in \Gamma$ and $u_n = p_n(D)u$ for $n \ge 0$. We may and do suppose that the norm on the Sobolev space W^s is defined by

$$\|u\|_{W^s} := \sum_{n \ge 0} 2^{2sn} \|u_n\|^2.$$

Set $\widetilde{n}(\gamma) = \max\{n(\gamma), m(\gamma) + (n(\gamma)/2)\}$ for $\gamma \in \Gamma$. Then there exists a constant $c > 0$ such that if $|\widetilde{n}(\gamma) - n| > c$, we have

$$d(\operatorname{supp}(\psi_\gamma), \operatorname{supp}(\tilde{p}_n)) > 2^{\max\{n, \widetilde{n}(\gamma)\} - c}.$$

We first prove $W^s(\mathbb{D}) \subset \mathscr{B}^\beta_\nu$ for $s > \beta$ and $\nu \ge 2d + 2$ by showing $\|u\|_{\beta,\nu} \le C\|u\|_{W^s}$ for $u \in C^\infty(\mathbb{D})$. For each $n \ge 0$, we have

$$\sum_{\gamma : \widetilde{n}(\gamma) = n} \|d^\nu_\gamma \cdot u_\gamma\|^2_{L^2} = \sum_{\gamma : \widetilde{n}(\gamma) = n} \left\| d^\nu_\gamma \cdot \sum_{n'=0}^\infty p_\gamma(x, D)^* \tilde{p}_{n'}(D) u_n \right\|^2_{L^2}.$$

We regard the operator $u \mapsto d_\gamma \cdot p_\gamma(x,D)^* p_{n'}(D)u$ as an integral operator with the kernel

$$\kappa_{n',\gamma}(x,x') = \frac{1}{(2\pi)^{2(2d+1)}} \int d_\gamma(x') e^{i\langle \xi, x'-y\rangle + i\langle \eta, y-x\rangle} \rho_\gamma(y)\psi_\gamma(\xi) p_{n'}(\eta) dy d\xi d\eta.$$

Fix some $\mu > \max\{2d+2, s\}$. Similarly to Lemma 7.3, we have

$$\begin{aligned}|\kappa_{n',\gamma}(x,x')| &\le C \int_{Z(\gamma)} d_\gamma^\nu(x') b_\gamma^{\mu+\nu}(x'-y) b_{n,0}^\mu(y-x) dy \\ &\le C \int_{Z(\gamma)} b_\gamma^\mu(x'-y) b_{n,0}^\mu(y-x) dy.\end{aligned}$$

Further, in the case $|n' - \widetilde{n}(\gamma)| > c$, we can show

$$|\kappa_{\gamma,n'}(x,x')| \le C 2^{-\mu \max\{n', \widetilde{n}(\gamma)\}/2} \int_{Z(\gamma)} b_\gamma^\mu(x'-y) b_{n,0}^\mu(y-x) dy,$$

applying the formula (9) of integration by part along a set of vectors $\{v_j\}_{j=0}^{2d}$ that form an orthogonal basis of E for μ times to the integral with respect y in $\kappa_{n',\gamma}(x,x')$. Therefore we obtain, by using Young inequality, that

$$\begin{aligned}\sum_{\gamma:\widetilde{n}(\gamma)=n} \|d_\gamma^\nu u_\gamma\|_{L^2}^2 \le & Cn^2 \sum_{n':|n'-n|\le c} \|u_{n'}\|_{L^2}^2 \\ & + Cn^2 \sum_{n':|n'-n|>c} 2^{-\mu\max\{n',n\}/2} \|u_{n'}\|_{L^2}^2.\end{aligned}$$

Take the sum on the both sides above with respect to n with weight $2^{\beta n}$. Then the sum on the left hand side is not smaller than $\|u\|_{\beta,\nu}$ and the sum on the right hand side is bounded by $C\|u\|_{W^s}$.

We next prove $\mathscr{B}_\nu^\beta \subset W^{-s}(\mathbb{D})$ for $s > \beta$ and $\nu \ge 2d+2$ by showing $\|u\|_{W^{-s}} \le C\|u\|_{\beta,\nu}$. We have

$$\|u_n\|_{L^2}^2 = \left\| p_n(D) \left(\sum_{|\widetilde{n}(\gamma)-n|<c} u_\gamma \right) \right\|_{L^2}^2 \le C \left\| \sum_{|\widetilde{n}(\gamma)-n|<c} u_\gamma \right\|_{L^2}^2$$

Since we have

$$|(u_\gamma, u_{\gamma'})| \le C_* \langle 2^{n/2}(z(\gamma) - z(\gamma'))\rangle^{-2\nu} \|(d_\gamma)^\nu u_\gamma\|_{L^2} \|(d_{\gamma'})^\nu u_{\gamma'}\|_{L^2}$$

for any pair $(\gamma,\gamma') \in \Gamma \times \Gamma$ and since the left hand side above vanishes if the supports of ψ_γ and $\psi_{\gamma'}$ does not meet, we obtain

$$\left\| \sum_{|\widetilde{n}(\gamma)-n|<c} u_\gamma \right\|_{L^2}^2 \le C \sum_{|\widetilde{n}(\gamma)-n|<c} \|u_\gamma\|_{L^2}^2$$

Take sum on the both sides above with respect to n with weight 2^{-sn}. Then the sum on the left hand side is not smaller than $C^{-1}\|u\|_{\beta,\nu}$ and that on the right hand side is bounded by $C\|u\|_{W^{-s}}$, provided $s > \beta$.

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